

THE BORDERLINES OF THE INVISIBILITY AND VISIBILITY FOR CALDERÓN'S INVERSE PROBLEM

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ABSTRACT. We consider the determination of a conductivity function in a two-dimensional domain from the Cauchy data of the solutions of the conductivity equation on the boundary. We prove uniqueness results for this inverse problem, posed by Calderón, for conductivities that are degenerate, that is, they may not be bounded from above or below. Elliptic equations with such coefficient functions are essential for physical models used in transformation optics and the metamaterial constructions. In particular, for scalar conductivities we solve the inverse problem in a class which is larger than L^∞ . Also, we give new counterexamples for the uniqueness of the inverse conductivity problem.

We say that a conductivity is visible if the inverse problem is solvable so that the inside of the domain can be uniquely determined, up to a change of coordinates, using the boundary measurements. The present counterexamples for the inverse problem have been related to the invisibility cloaking. This means that there are conductivities for which a part of the domain is shielded from detection via boundary measurements. Such conductivities are called invisibility cloaks.

In the present paper we identify the borderline of the visible conductivities and the borderline of invisibility cloaking conductivities. Surprisingly, these borderlines are not the same. We show that between the visible and the cloaking conductivities there are the electric holograms, conductivities which create an illusion of a non-existing body. The electric holograms give counterexamples for the uniqueness of the inverse problem which are less degenerate than the previously known ones. These examples are constructed using transformation optics and the inverse maps of the Iwaniec-Martin mappings. The uniqueness results are based on combining the complex geometrical optics, the properties of the mappings with subexponentially integrable distortion, and the Orlicz space techniques.

1. INTRODUCTION AND MAIN RESULTS

Invisibility cloaking has been a very topical subject in recent studies in mathematics, physics, and material science [2, 20, 28, 50, 44, 51, 57, 62]. By invisibility cloaking we mean the possibility, both theoretical and practical, of shielding a region or object from detection via electromagnetic fields.

The counterexamples for inverse problems and the proposals for invisibility cloaking are closely related. In 2003, before the appearance of practical possibilities for

cloaking, it was shown in [27, 28] that passive objects can be coated with a layer of material with a degenerate conductivity which makes the object undetectable by the electrostatic boundary measurements. These constructions were based on the blow up maps and gave counterexamples for the uniqueness of inverse conductivity problem in the three and higher dimensional cases. In two dimensional case, the mathematical theory of the cloaking examples for conductivity equation have been studied in [36, 37, 42, 54].

The interest in cloaking was raised in particular in 2006 when it was realized that practical cloaking constructions are possible using so-called metamaterials which allow fairly arbitrary specification of electromagnetic material parameters. The construction of Leonhardt [44] was based on conformal mapping on a non-trivial Riemannian surface. At the same time, Pendry et al [57] proposed a cloaking construction for Maxwell's equations using a blow up map and the idea was demonstrated in laboratory experiments [58]. There are also other suggestions for cloaking based on active sources [51] or negative material parameters [2, 50].

In this paper we consider both new counterexamples and uniqueness results for inverse problems. We study Calderón's inverse problem in the two dimensional case, that is, the question whether an unknown conductivity distribution inside a domain can be determined from the voltage and current measurements made on the boundary. Mathematically the measurements correspond to the knowledge of the Dirichlet-to-Neumann map Λ_σ (or the quadratic form) associated to σ , i.e., the map taking the Dirichlet boundary values of the solution of the conductivity equation

$$\nabla \cdot \sigma(x) \nabla u(x) = 0 \quad (1.1)$$

to the corresponding Neumann boundary values,

$$\Lambda_\sigma : u|_{\partial\Omega} \mapsto \nu \cdot \sigma \nabla u|_{\partial\Omega}. \quad (1.2)$$

In the classical theory of the problem, the conductivity σ is bounded uniformly from above and below. The problem was originally proposed by Calderón [15] in 1980. Sylvester and Uhlmann [60] proved the unique identifiability of the conductivity in dimensions three and higher for isotropic conductivities which are C^∞ -smooth, and Nachman gave a reconstruction method [52]. In three dimensions or higher unique identifiability of the conductivity is known for conductivities with $3/2$ derivatives [55], [12] and $C^{1,\alpha}$ -smooth conductivities which are C^∞ smooth outside surfaces on which they have conormal singularities [26]. The problems has also been solved with measurements only on a part of the boundary [34].

In two dimensions the first global solution of the inverse conductivity problem is due to Nachman [53] for conductivities with two derivatives. In this seminal paper the $\bar{\partial}$ technique was first time used in the study of Calderon's inverse problem. The smoothness requirements were reduced in [14] to Lipschitz conductivities. Finally, in [7] the uniqueness of the inverse problem was proven in the form that the

problem was originally formulated in [15], i.e., for general isotropic conductivities in L^∞ which are bounded from below and above by positive constants.

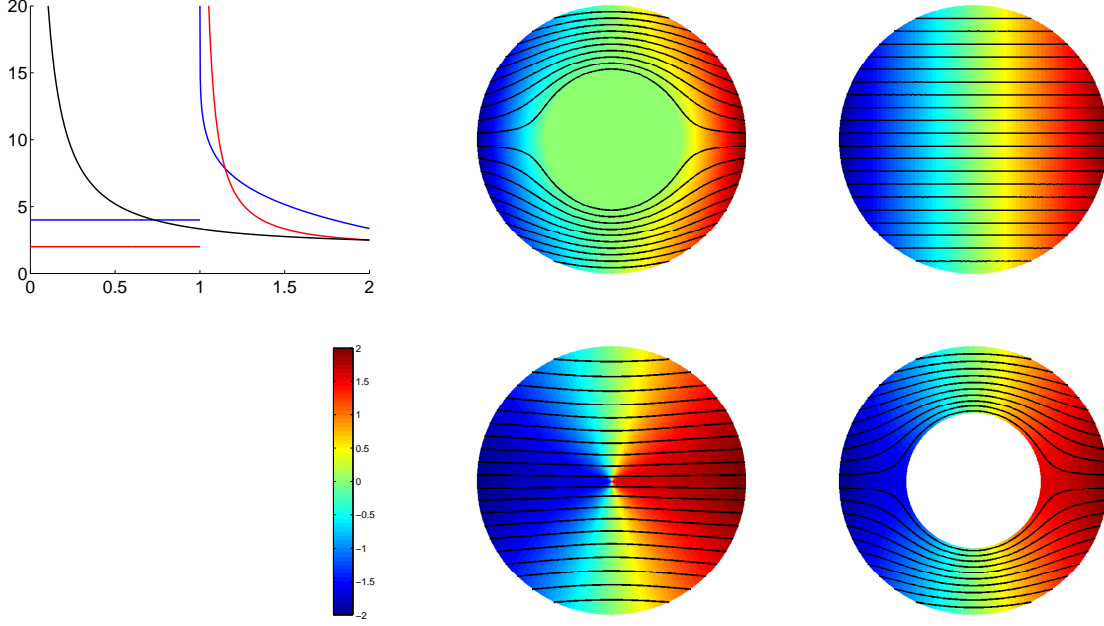


Figure 1. **Left.** $\text{tr}(\sigma)$ of three radial and singular conductivities on the positive x axis. The curves correspond to the invisibility cloaking conductivity (red), with the singularity $\sigma^{22}(x, 0) \sim (|x| - 1)^{-1}$ for $|x| > 1$, a visible conductivity (blue) with a log log type singularity at $|x| = 1$, and an electric hologram (black) with the conductivity having the singularity $\sigma^{11}(x, 0) \sim |x|^{-1}$. **Right, Top.** All measurements on the boundary of the invisibility cloak (left) coincide with the measurements for the homogeneous disc (right). The color shows the value of the solution u with the boundary value $u(x, y)|_{\partial B(2)} = x$ and the black curves are the integral curves of the current $-\sigma \nabla u$. **Right, Bottom.** All measurements on the boundary of the electric hologram (left) coincide with the measurements for an isolating disc covered with the homogeneous medium (right). The solutions and the current lines corresponding to the boundary value $u|_{\partial B(2)} = x$ are shown.

The Calderón problem with an anisotropic, i.e., matrix-valued, conductivity that is uniformly bounded from above and below has been studied in two dimensions [59, 53, 41, 8, 30] and in dimensions $n \geq 3$ [43, 41, 56]. For example, for the anisotropic inverse conductivity problem in the two dimensional case it is known that the Dirichlet-to-Neumann map determines a regular conductivity tensor up to a diffeomorphism $F : \overline{\Omega} \rightarrow \overline{\Omega}$, i.e., one can obtain an image of the interior of Ω in deformed coordinates. This implies that the inverse problem is not uniquely solvable, but the non-uniqueness of the problem can be characterized. We note

that the problem in higher dimensions is presently solved only in special cases, like when the conductivity is real analytic.

In this work we will study the inverse conductivity problem in the two dimensional case with degenerate conductivities. Such conductivities appear in physical models where the medium varies continuously from a perfect conductor to a perfect insulator. As an example, we may consider a case where the conductivity goes to zero or to infinity near ∂D where $D \subset \Omega$ is a smooth open set. We ask what kind of degeneracy prevents solving the inverse problem, that is, we study what is the border of visibility. We also ask what kind of degeneracy makes it even possible to coat of an arbitrary object so that it appears the same as a homogeneous body in all static measurements, that is, we study what is the border of the invisibility cloaking. Surprisingly, these borders are not the same; We identify these borderlines and show that between them there are the electric holograms, that is, the conductivities creating an illusion of a non-existing body (see Fig. 1). These conductivities are the counterexamples for the unique solvability of inverse problems for which even the topology of the domain can not be determined using boundary measurements. Our main result for the uniqueness of the inverse problem are given in Theorems 1.8, 1.9, and 1.11 and the counterexamples are formulated in Theorems 1.6 and 1.7.

The cloaking constructions have given rise for the design technique called the transformation optics. The metamaterials build to operate at microwave frequencies [58] and near the optical frequencies [19, 61] are inherently prone to dispersion, so that realistic cloaking must currently be considered as occurring at very narrow range of wavelengths. Fortunately, in many physical applications the materials need to operate only near a single frequency. In particular, the cloaking type constructions have inspired suggestions for possible devices producing extreme effects on wave propagation, including invisibility cloaks for magneto-statics [63], acoustics [18, 16] and quantum mechanics [65, 24]; field rotators [17]; electromagnetic wormholes [23]; invisible sensors [3, 22]; superantennas [46]; perfect absorbers [40]; and wave amplifiers [25]. It has turned out that the designs that are based on well posed mathematical models, e.g. approximate cloaks, have excellent properties when compared to ad hoc constructions. Due to this, it is important to know what is the exact degree of non-regularity which is needed for invisibility cloaking or solving the inverse problems.

Finally, we note that the differential equations with degenerate coefficients modeling cloaking devices have turned out to have interesting properties, such as non-existence results for solutions with non-zero sources [20, 47] and the local and non-local hidden boundary conditions [42, 54].

The structure of the paper is the following. The main results and the formulation of the boundary measurements are presented in the first section. The proofs for the existence of the solutions of the direct problem as well as for the

new counterexamples and the invisibility cloaking examples with a non-smooth background are given in Section 2. The uniqueness result for the isotropic conductivities is proven in Sections 3-4 and the reduction of the general problem to the isotropic case is shown in Section 5. In Sections 3-5, the degeneracy of the conductivity causes that the exponentially growing solutions, the standard tools used to study Calderon's inverse problem, can not be constructed using purely microlocal or functional analytic methods. Because of this we will extensively need the topological properties of the solutions: By Stoilow's theorem the solutions are compositions of analytic functions and homeomorphisms. Using this, the continuity properties of the weakly monotone maps, and the Orlicz-estimates holding for homeomorphisms we prove the existence of the solutions in the Sobolev-Orlicz spaces. These spaces are chosen so that we can obtain subexponential asymptotics for the families of exponentially growing solutions needed in the $\bar{\partial}$ technique used to solve the inverse problem.

1.1. Definition of measurements and solvability. Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain with a smooth boundary. Let $\Sigma = \Sigma(\Omega)$ be the class of measurable matrix valued functions $\sigma : \Omega \rightarrow M$, where M is the set of generalized matrices m of the form

$$m = U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U^t$$

where $U \in \mathbb{R}^{2 \times 2}$ is an orthogonal matrix, $U^{-1} = U^t$ and $\lambda_1, \lambda_2 \in [0, \infty)$. We denote by $W^{s,p}(\Omega)$ and $H^s(\Omega) = W^{s,2}(\Omega)$ the standard Sobolev spaces.

In the following, let $dm(z)$ denote the Lebesgue measure in \mathbb{C} and $|E|$ be the Lebesgue measure of the set $E \subset \mathbb{C}$. Instead of defining the Dirichlet-to-Neumann operator for the above conductivities, we consider the corresponding quadratic forms.

Definition 1.1. Let $h \in H^{1/2}(\partial\Omega)$. The Dirichlet-to-Neumann quadratic form corresponding to the conductivity $\sigma \in \Sigma(\Omega)$ is given by

$$Q_\sigma[h] = \inf A_\sigma[u] \quad \text{where, } A_\sigma[u] = \int_\Omega \sigma(z) \nabla u(z) \cdot \nabla u(z) \, dm(z), \quad (1.3)$$

and the infimum is taken over real valued $u \in L^1(\Omega)$ such that $\nabla u \in L^1(\Omega)^3$ and $u|_{\partial\Omega} = h$. In the case where $Q_\sigma[h] < \infty$ and $A_\sigma[u]$ reaches its minimum at some u , we say that u is a $W^{1,1}(\Omega)$ solution of the conductivity problem.

In the case when σ is smooth, bounded from below and above by positive constants, $Q_\sigma[h]$ is the quadratic form corresponding the Dirichlet-to-Neumann map (1.2),

$$Q_\sigma[h] = \int_{\partial\Omega} h \Lambda_\sigma h \, dS, \quad (1.4)$$

where dS is the length measure on $\partial\Omega$. Physically, $Q_\sigma[h]$ corresponds to the power needed to keep voltage h at the boundary. For smooth conductivities bounded from below, for every $h \in H^{1/2}(\partial\Omega)$ the integral $A_\sigma[u]$ always has a unique minimizer $u \in H^1(\Omega)$ with $u|_{\partial\Omega} = h$, which is also a distributional solution to (1.1). Conversely, for functions $u \in H^1(\Omega)$ their traces lie in $H^{1/2}(\partial\Omega)$. It is for this reason that we chose to consider the $H^{1/2}$ -boundary functions also in the most general case. We interpret that the Dirichlet-to-Neumann form corresponds to the idealization of the boundary measurements for $\sigma \in \Sigma(\Omega)$.

We note that the conductivities studied in the context of cloaking are not even in L^1_{loc} . As σ is unbounded it is possible that $Q_\sigma[h] = \infty$. Even if $Q_\sigma[h]$ is finite, the minimization problem in (1.3) may generally have no minimizer and even if they exist the minimizers need not be distributional solutions to (1.1). However, if the singularities of σ are not too strong, minimizers satisfying (1.1) do always exist. To show this, we need to define a suitable subclasses of degenerate conductivities.

Let $\sigma \in \Sigma(\Omega)$. We start with precise quantities describing the possible degeneracy or loss of uniform ellipticity. First, a natural measure of the anisotropy of the conductivity σ at $z \in \Omega$ is

$$k_\sigma(z) = \sqrt{\frac{\lambda_1(z)}{\lambda_2(z)}},$$

where $\lambda_1(z)$ and $\lambda_2(z)$ are the eigenvalues of the matrix $\sigma(z)$, $\lambda_1(z) \geq \lambda_2(z)$. If we want to simultaneously control both the size and the anisotropy, this is measured by the ellipticity $K(z) = K_\sigma(z)$ of $\sigma(z)$, i.e. the smallest number $1 \leq K(z) \leq \infty$ such that

$$\frac{1}{K(z)}|\xi|^2 \leq \xi \cdot \sigma(z)\xi \leq K(z)|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^2. \quad (1.5)$$

For a general, positive matrix valued function $\sigma(z)$ we have at $z \in \Omega$

$$K(z) = k_\sigma(z) \max\{[\det \sigma(z)]^{1/2}, [\det \sigma(z)]^{-1/2}\}. \quad (1.6)$$

Consequently, we always have the following simple estimates.

Lemma 1.2. *For any measurable matrix function $\sigma(z)$ we have*

$$\frac{1}{4} [\operatorname{tr} \sigma(z) + \operatorname{tr} (\sigma(z)^{-1})] \leq K(z) \leq \operatorname{tr} \sigma(z) + \operatorname{tr} (\sigma(z)^{-1}).$$

Proof. Let λ_{max} and λ_{min} be the eigenvalues of $\sigma = \sigma(z)$. Then $K(z) = \max(\lambda_{max}, \lambda_{min}^{-1})$. Since $\operatorname{tr} \sigma(z) = \lambda_{max} + \lambda_{min}$ and $\operatorname{tr} (\sigma(z)^{-1}) = \lambda_{max}^{-1} + \lambda_{min}^{-1}$, the claim follows. \square

Due to Lemma 1.2 we use the quantity $\operatorname{tr} \sigma(z) + \operatorname{tr} (\sigma(z)^{-1})$ as a measure of size and anisotropy of $\sigma(z)$.

For the degenerate elliptic equations it may be that the optimization problem (1.3) has a minimizer which satisfies the conductivity equation but this solution may not have the standard $W_{loc}^{1,2}$ regularity. Therefore more subtle smoothness estimates are required. We start with the exponentially integrable conductivities, and the natural energy estimates they require. As an important consequence we will see the correct Orlicz-Sobolev regularity to work with. These observations are based on the following elementary inequality.

Lemma 1.3. *Let $K \geq 1$ and $A \in \mathbb{R}^{2 \times 2}$ be a symmetric matrix satisfying*

$$\frac{1}{K}|\xi|^2 \leq \xi \cdot A\xi \leq K|\xi|^2, \quad \xi \in \mathbb{R}^2.$$

Then for every $p > 0$

$$\frac{|\xi|^2}{\log(e + |\xi|^2)} + \frac{|A\xi|^2}{\log(e + |A\xi|^2)} \leq \frac{2}{p} (\xi \cdot A\xi + e^{pK}).$$

Proof. Since $K \geq 1$ and $t \mapsto t/\log(e + t)$ is an increasing function, we have

$$\begin{aligned} \frac{|\xi|^2}{\log(e + |\xi|^2)} &\leq \frac{K\xi \cdot A\xi}{\log(e + K\xi \cdot A\xi)} \\ &\leq \frac{1}{p} \left(\frac{\xi \cdot A\xi}{\log(e + \xi \cdot A\xi)} \right) pK \\ &\leq \frac{1}{p} (\xi \cdot A\xi + e^{pK}), \end{aligned}$$

where the last estimate follows from the inequality

$$ab \leq a \log(e + a) + e^b, \quad a, b \geq 0.$$

Moreover, as K is at least as large as the maximal eigenvalue of A , we have $|A\xi|^2 \leq K\xi \cdot A\xi$. Thus we see as above that

$$\frac{|A\xi|^2}{\log(e + |A\xi|^2)} \leq \frac{K\xi \cdot A\xi}{\log(e + K\xi \cdot A\xi)} \leq \frac{1}{p} (\xi \cdot A\xi + e^{pK}).$$

Adding the above estimates together proves the claim. \square

Lemma 1.3 implies in particular that if $\sigma(z)$ is symmetric matrix valued function satisfying (1.5) for a.e. $z \in \Omega$ and $u \in W^{1,1}(\Omega)$, then always

(1.7)

$$\begin{aligned} p \int_{\Omega} \frac{|\nabla u(z)|^2}{\log(e + |\nabla u(z)|^2)} dm(z) &\leq \int_{\Omega} \nabla u(z) \cdot \sigma(z) \nabla u(z) dm(z) + \int_{\Omega} e^{pK(z)} dm(z), \\ p \int_{\Omega} \frac{|\sigma(z) \nabla u(z)|^2}{\log(e + |\sigma(z) \nabla u(z)|^2)} dm(z) &\leq \int_{\Omega} \nabla u(z) \cdot \sigma(z) \nabla u(z) dm(z) + \int_{\Omega} e^{pK(z)} dm(z). \end{aligned}$$

Note that these inequalities are valid whether u is a solution of the conductivity equation or not!

Due to (1.7), we see that to analyze finite energy solutions corresponding to a singular conductivity of exponentially integrable ellipticity, we are naturally led to consider the regularity gauge

$$Q(t) = \frac{t^2}{\log(e+t)}, \quad t \geq 0. \quad (1.8)$$

We say accordingly that f belongs to the Orlicz space $W^{1,Q}(\Omega)$, cf. Appendix, if f and its first distributional derivatives are in $L^1(\Omega)$ and

$$\int_{\Omega} \frac{|\nabla f(z)|^2}{\log(e + |\nabla f(z)|)} \, dm(z) < \infty.$$

The first existence result for solutions corresponding to degenerate conductivities is now given as follows.

Theorem 1.4. *Let $\sigma(z)$ be a measurable symmetric matrix valued function. Suppose further that for some $p > 0$,*

$$\int_{\Omega} \exp(p[tr \sigma(z) + tr(\sigma(z)^{-1})]) \, dm(z) = C_1 < \infty. \quad (1.9)$$

Then, if $h \in H^{1/2}(\partial\Omega)$ is such that $Q_{\sigma}[h] < \infty$ and $X = \{v \in W^{1,1}(\Omega); v|_{\partial\Omega} = h\}$, there is a unique $w \in X$ such that

$$A_{\sigma}[w] = \inf\{A_{\sigma}[v] ; v \in X\}. \quad (1.10)$$

Moreover, w satisfies the conductivity equation

$$\nabla \cdot \sigma \nabla w = 0 \quad \text{in } \Omega \quad (1.11)$$

in sense of distributions, and it has the regularity $w \in W^{1,Q}(\Omega) \cap C(\Omega)$.

Note that if σ is bounded near $\partial\Omega$ then $Q_{\sigma}[h] < \infty$ for all $h \in H^{1/2}(\partial\Omega)$. Theorem 1.4 is proven in Theorem 2.1 and Corollary 2.3 in a more general setting.

Theorem 1.4 yields that for conductivities satisfying (1.9) and being 1 near $\partial\Omega$ we can define the Dirichlet-to-Neumann map

$$\Lambda_{\sigma} : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega), \quad \Lambda_{\sigma}(u|_{\partial\Omega}) = \nu \cdot \sigma \nabla u|_{\partial\Omega}, \quad (1.12)$$

where u satisfies (1.1).

The reader should consider the exponential condition (1.9) as being close to the optimal one, still allowing uniqueness in the inverse problem. Indeed, in view of Theorem 1.7 and Section 1.5 below, the most general situation where the Calderón inverse problem can be solved involves conductivities whose singularities satisfy a physically interesting small relaxation of the condition (1.9). Before solving inverse problems for conductivities satisfying (1.9) we discuss some counterexamples.

1.2. Counterexamples for the unique solvability of the inverse problem.

Let $F : \Omega_1 \rightarrow \Omega_2$, $y = F(x)$ be an orientation preserving homeomorphism between domains $\Omega_1, \Omega_2 \subset \mathbb{C}$ for which F and its inverse F^{-1} are at least $W^{1,1}$ -smooth and let $\sigma(x) = [\sigma^{jk}(x)]_{j,k=1}^2 \in \Sigma(\Omega_1)$ be a conductivity on Ω_1 . Then the map F pushes σ forward to a conductivity $(F_*\sigma)(y)$, defined on Ω_2 and given by

$$(F_*\sigma)(y) = \frac{1}{[\det DF(x)]} DF(x) \sigma(x) DF(x)^t, \quad x = F^{-1}(y). \quad (1.13)$$

The main methods for constructing counterexamples to Calderón's problem are based on the following principle.

Proposition 1.5. *Assume that $\sigma, \tilde{\sigma} \in \Sigma(\Omega)$ satisfy (1.9), and let $F : \Omega \rightarrow \tilde{\Omega}$ be a homeomorphism so that F and F^{-1} are $W^{1,Q}$ -smooth and C^1 -smooth near the boundary, and $F|_{\partial\Omega} = id$. Suppose that $\tilde{\sigma} = F_*\sigma$. Then $Q_\sigma = Q_{\tilde{\sigma}}$.*

This proposition generalizes the previously known results [38] to less smooth diffeomorphisms and conductivities and it follows from Lemma 2.4 proven later.

1.3. Counterexample 1: Invisibility cloaking. We consider here invisibility cloaking in general background σ , that is, we aim to coat an arbitrary body with a layer of exotic material so that the coated body appears in measurements the same as the background conductivity σ . Usually one is interested in the case when the background conductivity σ is equal to the constant $\gamma = 1$. However, we consider here a more general case and assume that σ is a L^∞ -smooth conductivity in $\overline{B}(2)$, $\sigma(z) \geq c_0 I$, $c_0 > 0$. Here, $B(\rho)$ is an open 2-dimensional disc of radius ρ and center zero and $\overline{B}(\rho)$ is its closure. Consider a homeomorphism

$$F : \overline{B}(2) \setminus \{0\} \rightarrow \overline{B}(2) \setminus \mathcal{K} \quad (1.14)$$

where $\mathcal{K} \subset B(2)$ is a compact set which is the closure of a smooth open set and suppose $F : \overline{B}(2) \setminus \{0\} \rightarrow \overline{B}(2) \setminus \mathcal{K}$ and its inverse F^{-1} are C^1 -smooth in $\overline{B}(2) \setminus \{0\}$ and $\overline{B}(2) \setminus \mathcal{K}$, correspondingly. We also require that $F(z) = z$ for $z \in \partial B(2)$. The standard example of invisibility cloaking is the case when $\mathcal{K} = \overline{B}(1)$ and the map

$$F_0(z) = \left(\frac{|z|}{2} + 1\right) \frac{z}{|z|}. \quad (1.15)$$

Using the map (1.14), we define a singular conductivity

$$\tilde{\sigma}(z) = \begin{cases} (F_*\sigma)(z) & \text{for } z \in B(2) \setminus \mathcal{K}, \\ \eta(z) & \text{for } z \in \mathcal{K}, \end{cases} \quad (1.16)$$

where $\eta(z) = [\eta^{jk}(x)]$ is any symmetric measurable matrix satisfying $c_1 I \leq \eta(z) \leq c_2 I$ with $c_1, c_2 > 0$. The conductivity $\tilde{\sigma}$ is called the cloaking conductivity obtained from the transformation map F and background conductivity σ and $\eta(z)$ is the conductivity of the cloaked (i.e. hidden) object.

In particular, choosing σ to be the constant conductivity $\sigma = 1$, $\mathcal{K} = \overline{B}(1)$, and F to be the map F_0 given in (1.15), we obtain the standard example of the invisibility cloaking. In dimensions $n \geq 3$ it is shown in 2003 in [27, 28] that the Dirichlet-to-Neumann map corresponding to $H^1(\Omega)$ solutions for the conductivity (1.16) coincide with the Dirichlet-to-Neumann map for $\sigma = 1$. In 2008, the analogous result was proven in the two-dimensional case in [36]. For cloaking results for the Helmholtz equation with frequency $k \neq 0$ and for Maxwell's system in dimensions $n \geq 3$, see results in [20]. We note also that John Ball [11] has used the push forward by the analogous radial blow-up maps to study the discontinuity of the solutions of partial differential equations, in particular the appearance of cavitation in the non-linear elasticity.

In the sequel we consider cloaking results using measurements given in Definition 1.1. As we have formulated the boundary measurements in a new way, that is, in terms of the Dirichlet-to-Neumann forms Q_σ associated to the class $W^{1,1}(\Omega)$, we present the complete proof of the following proposition in Subsection 2.4.

Theorem 1.6. *(i) Let $\sigma \in L^\infty(B(2))$ be a scalar conductivity, $\sigma(x) \geq c_0 > 0$, $\mathcal{K} \subset B(2)$ be a relatively compact open set with smooth boundary and $F : \overline{B}(2) \setminus \{0\} \rightarrow \overline{B}(2) \setminus \mathcal{K}$ be a homeomorphism. Assume that F and F^{-1} are C^1 -smooth in $\overline{B}(2) \setminus \{0\}$ and $\overline{B}(2) \setminus \mathcal{K}$, correspondingly and $F|_{\partial B(2)} = \text{id}$. Moreover, assume there is $C_0 > 0$ such that $\|DF^{-1}(x)\| \leq C_0$ for all $x \in \overline{B}(2) \setminus \mathcal{K}$. Let $\tilde{\sigma}$ be the conductivity defined in (1.16). Then the boundary measurements for $\tilde{\sigma}$ and σ coincide in the sense that $Q_{\tilde{\sigma}} = Q_\sigma$.*

(ii) Let $\tilde{\sigma}$ be a cloaking conductivity of the form (1.16) obtained from the transformation map F and the background conductivity σ where F and σ satisfy the conditions in (i). Then

$$\text{tr}(\tilde{\sigma}) \notin L^1(B(2) \setminus \mathcal{K}). \quad (1.17)$$

The result (1.17) is optimal in the following sense. When F is the map F_0 in (1.15) and $\sigma = 1$, the eigenvalues of the cloaking conductivity $\tilde{\sigma}$ in $B(2) \setminus \overline{B}(1)$ behaves asymptotically as $(|z| - 1)$ and $(|z| - 1)^{-1}$ as $|z| \rightarrow 1$. This cloaking conductivity has so strong degeneracy that (1.17) holds. On the other hand,

$$\text{tr}(\tilde{\sigma}) \in L^1_{\text{weak}}(B(2)). \quad (1.18)$$

where L^1_{weak} is the weak- L^1 space. We note that in the case when $\sigma = 1$, $\det(\tilde{\sigma})$ is identically 1 in $B(2) \setminus \overline{B}(1)$.

The formula (1.18) for the blow up map F_0 in (1.15) and Theorem 1.6 identify the *borderline of the invisibility* for the trace of the conductivity: Any cloaking conductivity $\tilde{\sigma}$ satisfies $\text{tr}(\tilde{\sigma}) \notin L^1(B(2))$ and there is an example of a cloaking conductivity for which $\text{tr}(\tilde{\sigma}) \in L^1_{\text{weak}}(B(2))$. Thus the borderline of invisibility is the same as the border between the space L^1 and the weak- L^1 space.

1.4. Counterexample 2: Illusion of a non-existent obstacle. Next we consider new counterexamples for the inverse problem which could be considered as creating an illusion of a non-existing obstacle. The example is based on a radial shrinking map, that is, a mapping $B(2) \setminus \overline{B}(1) \rightarrow B(2) \setminus \{0\}$. The suitable maps are the inverse maps of the blow-up maps $F_1 : B(2) \setminus \{0\} \rightarrow B(2) \setminus \overline{B}(1)$ which are constructed by Iwaniec and Martin [31] and have the optimal smoothness. Alternative constructions for such blow up maps have also been proposed by Kauhanen et al, see [33]. Using the properties of these maps and defining a conductivity $\sigma_1 = (F_1^{-1})_* 1$ on $B(2) \setminus \{0\}$ we will later prove the following result.

Theorem 1.7. *Let γ_1 be a conductivity in $B(2)$ which is identically 1 in $B(2) \setminus \overline{B}(1)$ and zero in $B(1)$ and $\mathcal{A} : [1, \infty] \rightarrow [0, \infty]$ be any strictly increasing positive smooth function with $\mathcal{A}(1) = 0$ which is sub-linear in the sense that*

$$\int_1^\infty \frac{\mathcal{A}(t)}{t^2} dt < \infty. \quad (1.19)$$

Then there is a conductivity $\sigma_1 \in \Sigma(B_2)$ satisfying $\det(\sigma_1) = 1$ and

$$\int_{B(2)} \exp(\mathcal{A}(\text{tr}(\sigma_1) + \text{tr}(\sigma_1^{-1}))) dm(z) < \infty, \quad (1.20)$$

such that $Q_{\sigma_1} = Q_{\gamma_1}$, i.e., the boundary measurements corresponding to σ_1 and γ_1 coincide.

We observe that for instance the function $\mathcal{A}_0(t) = t/(1 + \log t)^{1+\varepsilon}$ satisfies (1.19) and for such weight function $\sigma_1 \in L^1(B_2)$. The proof of Theorem 1.7 is given in Subsection 2.4.

Note that γ_1 corresponds to the case when $B(1)$ is a perfect insulator which is surrounded with constant conductivity 1. Thus Theorem 1.7 can be interpreted by saying that there is a relatively weakly degenerated conductivity satisfying integrability condition (1.20) that creates in the boundary observations an illusion of an obstacle that does not exist. Thus the conductivity can be considered as "electric hologram". As the obstacle can be considered as a "hole" in the domain, we can say also that even the topology of the domain can not be detected. In other words, Calderón's program to image the conductivity inside a domain using the boundary measurements cannot work within the class of degenerate conductivities satisfying (1.19) and (1.20).

1.5. Positive results for Calderón's inverse problem. Let us formulate our first main theorem which deals on inverse problems for anisotropic conductivities where both the trace and the determinant of the conductivity can be degenerate.

Theorem 1.8. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain with smooth boundary. Let $\sigma_1, \sigma_2 \in \Sigma(\Omega)$ be matrix valued conductivities in Ω which satisfy*

the integrability condition

$$\int_{\Omega} \exp(p(\operatorname{tr} \sigma(z) + \operatorname{tr}(\sigma(z)^{-1}))) dm(z) < \infty$$

for some $p > 1$. Moreover, assume that

$$\int_{\Omega} \mathcal{E}(q \det \sigma_j(z)) dm(z) < \infty, \quad \text{for some } q > 0, \quad (1.21)$$

where $\mathcal{E}(t) = \exp(\exp(\exp(t^{1/2} + t^{-1/2})))$ and $Q_{\sigma_1} = Q_{\sigma_2}$. Then there is a $W_{loc}^{1,1}$ -homeomorphism $F : \Omega \rightarrow \Omega$ satisfying $F|_{\partial\Omega} = \operatorname{id}$ such that

$$\sigma_1 = F_* \sigma_2. \quad (1.22)$$

Equation (1.22) can be stated as saying that σ_1 and σ_2 are the same up to a change of coordinates, that is, the invariant manifold structures corresponding to these conductivities are the same, cf. [43, 41].

In the case when the conductivities are isotropic we can improve the result of Theorem 1.8. The following theorem is our second main result for uniqueness of the inverse problem.

Theorem 1.9. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain with smooth boundary. If $\sigma_1, \sigma_2 \in \Sigma(\Omega)$ are isotropic conductivities, i.e., $\sigma_j(z) = \gamma_j(z)I$, $\gamma_j(z) \in [0, \infty]$ satisfying*

$$\int_{\Omega} \exp(\exp[q(\gamma_j(z) + \frac{1}{\gamma_j(z)})]) dm(z) < \infty, \quad \text{for some } q > 0, \quad (1.23)$$

and $Q_{\sigma_1} = Q_{\sigma_2}$, then $\sigma_1 = \sigma_2$.

Let us next consider anisotropic conductivities with bounded determinant but more degenerate ellipticity function $K_{\sigma}(z)$ defined in (1.5), and ask how far can we then generalize Theorem 1.8. Motivated by the counterexample given in Theorem 1.7 we consider the following class: We say that $\sigma \in \Sigma(\Omega)$ has an exponentially degenerated anisotropy with a weight \mathcal{A} and denote $\sigma \in \Sigma_{\mathcal{A}} = \Sigma_{\mathcal{A}}(\Omega)$ if $\sigma(z) \in \mathbb{R}^{2 \times 2}$ for a.e. $z \in \Omega$ and

$$\int_{\Omega} \exp(\mathcal{A}(\operatorname{tr} \sigma + \operatorname{tr}(\sigma^{-1}))) dm(z) < \infty. \quad (1.24)$$

In view of Theorem 1.7, for obtaining uniqueness for the inverse problem we need to consider weights that are strictly increasing positive smooth functions $\mathcal{A} : [1, \infty] \rightarrow [0, \infty]$, $\mathcal{A}(1) = 0$, with

$$\int_1^{\infty} \frac{\mathcal{A}(t)}{t^2} dt = \infty \quad \text{and} \quad t\mathcal{A}'(t) \rightarrow \infty, \quad \text{as } t \rightarrow \infty. \quad (1.25)$$

We say that \mathcal{A} has almost linear growth if (1.25) holds. The point here is the first condition, that is, the divergence of the integral. The second condition is a technicality, which is satisfied by all weights one encounters in practice (which do

not oscillate too much); the condition guarantees that the Sobolev-gauge function $P(t)$ defined below in (1.26) is equivalent to a convex function for large t , see [6, Lem. 20.5.4].

Note in particular that affine weights $\mathcal{A}(t) = pt - p$, $p > 0$ satisfy the condition (1.25). To develop uniqueness results for inverse problems within the class $\Sigma_{\mathcal{A}}$, the first questions we face are to establish the right Sobolev-Orlicz regularity for the solutions u of finite energy, $A_{\sigma}[u] < \infty$, and solving the Dirichlet problem with given boundary values.

To start with this, we need the counterpart of the gauge $Q(t)$ defined at (1.8). In the case of a general weight \mathcal{A} we define

$$P(t) = \begin{cases} t^2, & \text{for } 0 \leq t < 1, \\ \frac{t^2}{\mathcal{A}^{-1}(\log(t^2))}, & \text{for } t \geq 1 \end{cases} \quad (1.26)$$

where \mathcal{A}^{-1} is the inverse function of \mathcal{A} . As an example, note that if \mathcal{A} is affine, $\mathcal{A}(t) = pt - p$ for some number $p > 0$, then the condition (1.24) takes us back to the exponentially integrable distortion of Theorem 1.8, while $P(t) = t^2(1 + \frac{1}{p} \log^+(t^2))^{-1}$ is equivalent to the gauge function $Q(t)$ used at (1.8).

The inequalities (1.7) corresponding to the case when \mathcal{A} is affine can be generalized for the following result holding for general gauge \mathcal{A} satisfying (1.25).

Lemma 1.10. *Suppose $u \in W_{loc}^{1,1}(\Omega)$ and \mathcal{A} satisfies the almost linear growth condition (1.25). Then*

$$\int_{\Omega} (P(|\nabla u|) + P(|\sigma \nabla u|)) dm \leq 2 \int_{\Omega} e^{\mathcal{A}(\text{tr} \sigma + \text{tr}(\sigma^{-1}))} dm(z) + 2 \int_{\Omega} \nabla u \cdot \sigma \nabla u dm$$

for every measurable function of symmetric matrices $\sigma(z) \in \mathbb{R}^{2 \times 2}$.

Proof. We have in fact pointwise estimates. For these, note first that the conditions for $\mathcal{A}(t)$ imply that $P(t) \leq t^2$ for every $t \geq 0$. Hence, if $|\nabla u(z)|^2 \leq \exp \mathcal{A}(\text{tr} \sigma(z) + \text{tr}(\sigma^{-1}(z)))$ then

$$P(|\nabla u(z)|) \leq \exp \mathcal{A}(\text{tr} \sigma(z) + \text{tr}(\sigma^{-1}(z))). \quad (1.27)$$

If, however, $|\nabla u(z)|^2 > \exp \mathcal{A}(\text{tr} \sigma(z) + \text{tr}(\sigma^{-1}(z)))$, then

$$P(|\nabla u(z)|) = \frac{|\nabla u(z)|^2}{\mathcal{A}^{-1}(\log |\nabla u(z)|^2)} \leq \frac{|\nabla u(z)|^2}{\text{tr}(\sigma^{-1}(z))} \leq \nabla u(z) \cdot \sigma(z) \nabla u(z). \quad (1.28)$$

Thus at a.e. $z \in \Omega$ we have

$$P(|\nabla u(z)|) \leq \exp \mathcal{A}(\text{tr} \sigma(z) + \text{tr}(\sigma^{-1}(z))) + \nabla u(z) \cdot \sigma(z) \nabla u(z). \quad (1.29)$$

Similar arguments give pointwise bounds to $P(|\sigma(z) \nabla u(z)|)$. Summing these estimates and integrating these pointwise estimates over Ω proves the claim. \square

In following, we say that $u \in W_{loc}^{1,1}(\Omega)$ is in the Orlicz space $W^{1,P}(\Omega)$ if

$$\int_{\Omega} P(|\nabla u(z)|) dm(z) < \infty.$$

There are further important reasons that make the gauge $P(t)$ a natural and useful choice. For instance, in constructing a minimizer for the energy $A_{\sigma}[u]$ we are faced with the problem of possible equicontinuity of Sobolev functions with uniformly bounded $A_{\sigma}[u]$. In view of Lemma 1.10 this is reduced to describing those weight functions $\mathcal{A}(t)$ for which the condition $P(|\nabla u(z)|) \in L^1(\Omega)$ implies that the continuity modulus of u can be estimated. As we will see later in (3.12), this follows for weakly monotone functions u (in particular for homeomorphisms), as soon as the divergence condition

$$\int_1^{\infty} \frac{P(t)}{t^3} dt = \infty \quad (1.30)$$

is satisfied, that is, $P(t)$ has almost quadratic growth. In fact, note that the divergence of the integral $\int_1^{\infty} \frac{\mathcal{A}(t)}{t^2} dt$ is equivalent to

$$\int_1^{\infty} \frac{P(t)}{t^3} dt = \frac{1}{2} \int_1^{\infty} \frac{\mathcal{A}'(t)}{t} dt = \frac{1}{2} \int_1^{\infty} \frac{\mathcal{A}(t)}{t^2} dt = \infty \quad (1.31)$$

where we have used the substitution $\mathcal{A}(s) = \log(t^2)$. Thus the condition (1.25) is directly connected to the smoothness properties of solutions of finite energy, for conductivities satisfying (1.24).

We are now ready to formulate our third main theorem for uniqueness of inverse problem, which gives a sharp result for singular anisotropic conductivities with a determinant bounded from above and below by positive constants.

Theorem 1.11. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain with smooth boundary and $\mathcal{A} : [1, \infty) \rightarrow [0, \infty)$ be a strictly increasing smooth function satisfying the almost linear growth condition (1.25). Let $\sigma_1, \sigma_2 \in \Sigma(\Omega)$ be matrix valued conductivities in Ω which satisfy the integrability condition*

$$\int_{\Omega} \exp(\mathcal{A}(\text{tr} \sigma(z) + \text{tr}(\sigma(z)^{-1}))) dm(z) < \infty. \quad (1.32)$$

Moreover, suppose that $c_1 \leq \det(\sigma_j(z)) \leq c_2$, $z \in \Omega$, $j = 1, 2$ for some $c_1, c_2 > 0$ and $Q_{\sigma_1} = Q_{\sigma_2}$. Then there is a $W_{loc}^{1,1}$ -homeomorphism $F : \Omega \rightarrow \Omega$ satisfying $F|_{\partial\Omega} = \text{id}$ such that

$$\sigma_1 = F_* \sigma_2.$$

We note that the determination of σ from Q_{σ} in Theorems 1.8, 1.9, and 1.11 is constructive in the sense that one can write an algorithm which constructs σ from Λ_{σ} . For example, for the non-degenerate scalar conductivities such a construction has been numerically implemented in [9].

Let us next discuss the borderline of the visibility somewhat formally. Below we say that a conductivity is visible if there is an algorithm which reconstructs the conductivity σ from the boundary measurements Q_σ , possibly up to a change of coordinates. In other words, for visible conductivities one can use the boundary measurements to produce an image of the conductivity in the interior of Ω in some deformed coordinates. For simplicity, let us consider conductivities with $\det \sigma$ bounded from above and below. Then, Theorems 1.7 and 1.11 can be interpreted by saying that the almost linear growth condition (1.25) for the weight function \mathcal{A} gives the *borderline of visibility* for the trace of the conductivity matrix: If \mathcal{A} satisfies (1.25), the conductivities satisfying the integrability condition (1.32) are visible. However, if \mathcal{A} does not satisfy (1.25) we can construct a conductivity in Ω satisfying the integrability condition (1.32) which appears as if an obstacle (which does not exist in reality) would have included in the domain.

Thus the borderline of the visibility is between any spaces $\Sigma_{\mathcal{A}_1}$ and $\Sigma_{\mathcal{A}_2}$ where \mathcal{A}_1 satisfies condition (1.25) and \mathcal{A}_2 does not satisfy it. Example of such gauge functions are $\mathcal{A}_1(t) = t(1 + \log t)^{-1}$ and $\mathcal{A}_2(t) = t(1 + \log t)^{-1-\varepsilon}$ with $\varepsilon > 0$.

Summarizing the results, in terms of the trace of the conductivity, we have identified the borderline of visible conductivities and the borderline of invisibility cloaking conductivities. Moreover, these borderlines are not the same and between the visible and the invisibility cloaking conductivities there are conductivities creating electric holograms.

2. PROOFS FOR THE EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE DIRECT PROBLEM AND FOR THE COUNTEREXAMPLES.

First we show that under the conditions (1.24) and (1.25) the Dirichlet problem for the conductivity equation admits a unique solution u with finite energy $A_\sigma[u]$.

2.1. The Dirichlet problem. In this section we prove Theorem 1.4. In fact, we prove it in a more general setting than it was stated.

Theorem 2.1. *Let $\sigma \in \Sigma_{\mathcal{A}}(\Omega)$ where \mathcal{A} satisfies the almost linear growth condition (1.19). Then, if $h \in H^{1/2}(\partial\Omega)$ is such that $Q_\sigma[h] < \infty$ and $X = \{v \in W^{1,1}(\Omega); v|_{\partial\Omega} = h\}$, there is a unique $w \in X$ satisfies (1.10). Moreover, w satisfies the conductivity equation*

$$\nabla \cdot \sigma \nabla w = 0 \quad \text{in } \Omega \tag{2.1}$$

in sense of distributions, and it has the regularity $w \in W^{1,P}(\Omega)$.

Proof. For $N > 0$, denote $\Omega_N = \{x \in \Omega; \|\sigma(x)\| + \|\sigma(x)^{-1}\| \leq N\}$. Let $w_n \in X$ be such that

$$\lim_{n \rightarrow \infty} A_\sigma[w_n] = C_0 = \inf\{A_\sigma[v]; v \in X\} = Q_\sigma[h] < \infty$$

and $A_\sigma[w_n] < C_0 + 1$. Then by Lemma 1.10,

$$\int_{\Omega} P(|\nabla w_n(x)|) dm(x) + \int_{\Omega} P(|\sigma(x)\nabla w_n(x)|) dm(x) \leq 2(C_1 + C_0 + 1) = C_2, \quad (2.2)$$

where $C_1 = \int_{\Omega} e^{A(K(z))} dm(z)$. By [6, Lem. 20.5.3, 20.5.4], there is a convex, and unbounded function $\Phi : [0, \infty) \rightarrow \mathbb{R}$ such that $\Phi(t) \leq P(t) + c_0 \leq 2\Phi(t)$ with some $c_0 > 0$ and moreover, the function $t \mapsto \Phi(t^{5/8})$ is convex and increasing. This implies that $P(t) \geq c_1 t^{8/5} - c_2$ for some $c_1 > 0$, $c_2 \in \mathbb{R}$. Thus (2.2) yields that for all $1 < q \leq 8/5$

$$\|\nabla w_n\|_{L^q(\Omega)} \leq C_3 = C_3(q, C_0, C_1), \quad \text{for } n \in \mathbb{Z}_+.$$

Using the Poincare inequality in $L^q(\Omega)$ and that $(w_n - w_1)|_{\partial\Omega} = 0$, we see that $\|w_n - w_1\|_{L^q(\Omega)} \leq C_4 C_3$. Thus, there is C_5 such that $\|\nabla w_n\|_{W^{1,q}(\Omega)} < C_5$ for all n . By Banach-Alaoglu theorem this implies that by restricting to a subsequence of $(w_n)_{n=1}^\infty$, which we denote in sequel also by w_n , such that $w_n \rightarrow w$ in $W^{1,q}(\Omega)$ as $n \rightarrow \infty$. As $W^{1,q}(\Omega)$ embeds compactly to $H^s(\Omega)$ for $s < 2(1 - q^{-1})$ we see that $\|w_n - w\|_{H^s(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$ for all $s \in (\frac{1}{2}, \frac{3}{4})$. Thus $w_n|_{\partial\Omega} \rightarrow w|_{\partial\Omega}$ in $H^{s-1/2}(\partial\Omega)$ as $n \rightarrow \infty$. This implies that $w|_{\partial\Omega} = h$ and $w \in X$. Moreover, for any $N > 0$

$$\frac{1}{N} \int_{\Omega_N} |\nabla w_n(x)|^2 dm(x) \leq \int_{\Omega_N} \nabla w_n(x) \cdot \sigma(x) \nabla w_n(x) dm(x) \leq C_0 + 1.$$

This implies that $\nabla w_n|_{\Omega_N}$ are uniformly bounded in $L^2(\Omega_N)^2$. Thus by restricting to a subsequence, we can assume that $\nabla w_n|_{\Omega_N}$ converges weakly in $L^2(\Omega_N)^2$ as $n \rightarrow \infty$. Clearly, the weak limit must be $\nabla w|_{\Omega_N}$. Since the norm $V \mapsto (\int_{\Omega_N} V \cdot \sigma V dm)^{1/2}$ in $L^2(\Omega_N)^2$ is weakly lower semicontinuous, we see that

$$\int_{\Omega_N} \nabla w(x) \cdot \sigma(x) \nabla w(x) dm(x) \leq \liminf_{n \rightarrow \infty} \int_{\Omega_N} \nabla w_n(x) \cdot \sigma(x) \nabla w_n(x) dm(x) \leq C_0.$$

As this holds for all N , we see by applying the monotone convergence theorem as $N \rightarrow \infty$ that (1.10) holds. Thus w is a minimizer of A_σ in X .

By the above, $\sigma \nabla w_n \rightarrow \sigma \nabla w$ weakly in $L^2(\Omega_N)$ as $n \rightarrow \infty$ for all N . As noted above there is a convex function $\Phi : [0, \infty) \rightarrow \mathbb{R}$ such that $\Phi(t) \leq P(t) + c_0 \leq 2\Phi(t)$, $c_0 > 0$ and that $\Phi(t)$ is increasing for large values of t . Thus it follows from the semicontinuity results for integral operators, [10, Thm. 13.1.2], Lebesgue's monotone convergence theorem, and (2.2) that

$$\begin{aligned} \int_{\Omega} (\Phi(|\nabla w|) + \Phi(|\sigma \nabla w|)) dm(x) &\leq \lim_{N \rightarrow \infty} \liminf_{n \rightarrow \infty} \int_{\Omega_N} (\Phi(|\nabla w_n|) + \Phi(|\sigma \nabla w_n|)) dm \\ &\leq \lim_{N \rightarrow \infty} \liminf_{n \rightarrow \infty} \int_{\Omega_N} (P(|\nabla w_n|) + P(|\sigma \nabla w_n|)) dm + 2c_0|\Omega| \leq C_2 + 2c_0|\Omega|. \end{aligned}$$

It follows from the above and the inequality $P(t) \geq c_1 t^{8/5} - c_2$ that $\sigma(x) \nabla w(x) \in L^1(\Omega)$. Consider next $\phi \in C_0^\infty(\Omega)$. As $w + t\phi \in X$, $t \in \mathbb{R}$ and as w is a minimizer

of A_σ in X it follows that

$$\left. \frac{d}{dt} A_\sigma[w + t\phi] \right|_{t=0} = 2 \int_{\Omega} \nabla \phi(x) \cdot \sigma(x) \nabla w(x) dm(x) = 0.$$

This shows that the conductivity equation (2.1) is valid in the sense of distributions.

Next, assume that w and \tilde{w} are both minimizers of A_σ in X . Using the convexity of A_σ we see that then the second derivative of $t \mapsto A_\sigma[tw + (1-t)\tilde{w}]$ vanishes at $t = 0$. This implies that $\nabla(w - \tilde{w}) = 0$ for a.e. $x \in \Omega$. As w and \tilde{w} coincide at the boundary, this yields that $w = \tilde{w}$ and thus the minimizer is unique. \square

The fact that the minimizer w is continuous will be proven in the next subsection.

2.2. The Beltrami equation. It is natural to ask if the minimizer w in (1.10) is the only solution of finite σ -energy $A_\sigma[w]$ to the boundary value problem

$$\begin{aligned} \nabla \cdot \sigma \nabla w &= 0 \quad \text{in } \Omega, \\ w|_{\partial\Omega} &= h. \end{aligned} \tag{2.3}$$

It turns out that this is the case and to prove this we introduce one of the basic tools in this work, the Beltrami differential equation.

For this end, recall the Hodge-star operator $*$ which in two dimensions is just the rotation

$$* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that $\nabla \cdot (*\nabla u) = w$ for all $w \in W^{1,1}(\Omega)$ and recall that $\Omega \subset \mathbb{C}$ is simply connected. If $\sigma(x) = [\sigma^{jk}(x)]_{j,k=1}^2 \in \Sigma_{\mathcal{A}}(\Omega)$, where \mathcal{A} satisfies (1.19), and if $u \in W^{1,1}(\Omega)$ is a distributional solution to the conductivity equation

$$\nabla \cdot \sigma(x) \nabla u(x) = 0, \tag{2.4}$$

then by Lemma 1.10 we have $P(\nabla u), P(\sigma \nabla u) \in L^1(\Omega)$ and thus in particular $\sigma \nabla u \in L^1(\Omega)$. By (2.4) and the Poincaré lemma there is a function $v \in W^{1,1}(\Omega)$ such that

$$\nabla v = * \sigma(x) \nabla u(x). \tag{2.5}$$

Then

$$\nabla \cdot \sigma^*(x) \nabla v = 0 \quad \text{in } \Omega, \quad \sigma^*(x) = * \sigma(x)^{-1} *. \tag{2.6}$$

In particular, the above shows that $u, v \in W^{1,P}(\Omega)$. Moreover, an explicit calculation, see e.g. [6, formula (16.20)], reveals that the function $f = u + iv$ satisfies

$$\partial_{\bar{z}} f = \mu \partial_z f + \nu \overline{\partial_z f}, \tag{2.7}$$

where

$$\mu = \frac{\sigma^{22} - \sigma^{11} - 2i\sigma^{12}}{1 + \operatorname{tr}(\sigma) + \det(\sigma)}, \quad \nu = \frac{1 - \det(\sigma)}{1 + \operatorname{tr}(\sigma) + \det(\sigma)}, \quad (2.8)$$

and $\partial_{\bar{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$ with $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$. Summarizing, for $\sigma \in \Sigma_{\mathcal{A}}(\Omega)$ any distributional solution $u \in W^{1,1}(\Omega)$ of (2.4) is a real part of the solution f of (2.7). Conversely, the real part of any solution $f \in W^{1,1}(\Omega)$ of (2.7) satisfies (2.4) while the imaginary part is a solution to (2.6) and as $\sigma \in \Sigma_{\mathcal{A}}(\Omega)$, (2.4)-(2.6) and Lemma 1.10 yield that $u, v \in W^{1,P}(\Omega)$, and hence $f \in W^{1,P}(\Omega)$.

Furthermore, the ellipticity bound of $\sigma(z)$ is closely related to the distortion of the mapping f . Indeed, in the case when $\sigma(z_0) = \operatorname{diag}(\lambda_1, \lambda_2)$, a direct computation shows that

$$K_{\sigma}(z_0) = K_{\mu,\nu}(z_0), \quad \text{where } K_{\mu,\nu}(z) = \frac{1 + |\mu(z)| + |\nu(z)|}{1 - (|\mu(z)| + |\nu(z)|)} \quad (2.9)$$

and $K_{\sigma}(z)$ is the ellipticity of $\sigma(z)$ defined in (1.5). Using the chain rule for the complex derivatives, which can be written as

$$\partial(v \circ F) = (\partial v) \circ F \cdot \partial F + (\bar{\partial} v) \circ F \cdot \bar{\partial} F, \quad (2.10)$$

$$\bar{\partial}(v \circ F) = (\bar{\partial} v) \circ F \cdot \bar{\partial} F + (\partial v) \circ F \cdot \partial F, \quad (2.11)$$

we see that $|\mu(z)|$ and $|\nu(z)|$ do not change in an orthogonal rotation of the coordinate axis, $z \mapsto \alpha z$ where $\alpha \in \mathbb{C}$, $|\alpha| = 1$. As for any $z_0 \in \Omega$ there exists an orthogonal rotation of the coordinate axis so that matrix $\sigma(z_0)$ is diagonal in the rotated coordinates, we see that the identity (2.9) holds for all $z_0 \in \Omega$.

The equation (2.7) is also equivalent to the Beltrami equation

$$\bar{\partial} f(z) = \tilde{\mu}(z) \partial f(z) \quad \text{in } \Omega, \quad (2.12)$$

with the Beltrami coefficient

$$\tilde{\mu}(z) = \begin{cases} \mu(z) + \nu(z) \partial_z f(x) \left(\overline{\partial_z f(x)} \right)^{-1} & \text{if } \partial_z f(x) \neq 0, \\ \mu(z) & \text{if } \partial_z f(x) = 0, \end{cases} \quad (2.13)$$

satisfying $|\tilde{\mu}(z)| \leq |\mu(z)| + |\nu(z)|$ pointwise. We define the distortion of f at z be

$$K(z, f) := K_{\tilde{\mu}}(z) = \frac{1 + |\tilde{\mu}(z)|}{1 - |\tilde{\mu}(z)|} \leq K_{\sigma}(z), \quad z \in \Omega. \quad (2.14)$$

Below will also use the notation $K(z, f) = K_f(z)$.

In the sequel we will use frequently these different interpretations of the Beltrami equation. Note that $K(z, f) = (1 + |\tilde{\mu}(z)|)/(1 - |\tilde{\mu}(z)|)$ so that $K(z, f) = (|\partial f| + |\bar{\partial} f|)/(|\partial f| - |\bar{\partial} f|)$. As $\|Df\|^2 = (|\partial f| + |\bar{\partial} f|)^2$ and $J(z, f) = |\partial f|^2 - |\bar{\partial} f|^2$, this yields the distortion equality, see e.g. [6, formula (20.3)],

$$\|Df(z)\|^2 = K(z, f) J(z, f), \quad \text{for a.e. } z \in \Omega. \quad (2.15)$$

We will use extensively the fact that if a homeomorphism $F : \Omega \rightarrow \Omega'$, $F \in W^{1,1}(\Omega)$ is a finite distortion mapping with the distortion $K_F \in L^1(\Omega)$ then by [29] or [6, Thm. 21.1.4] the inverse function $H = F^{-1} : \Omega' \rightarrow \Omega$ is in $W^{1,2}(\Omega')$ and its derivative DH satisfies

$$\|DH\|_{L^2(\Omega')} \leq 2\|K_F\|_{L^1(\Omega)}. \quad (2.16)$$

We will also need few basic notions, see [6], from the theory of Beltrami equations. As the coefficients μ, ν are defined only in the bounded domain Ω , outside Ω we set $\mu(z) = \nu(z) = 0$ and $\sigma(z) = 1$, and consider global solutions to (2.7) in \mathbb{C} . In particular, we consider the case when Ω is the unit disc $\mathbb{D} = B(1)$. We say that a solution $f \in W_{loc}^{1,1}(\mathbb{C})$ of the equation (2.7) in $z \in \mathbb{C}$ is a *principal solution* if

1. $f : \mathbb{C} \rightarrow \mathbb{C}$ is a homeomorphism of \mathbb{C} and
2. $f(z) = z + \mathcal{O}(1/z)$ as $z \rightarrow \infty$.

The existence principal solutions is a fundamental fact that holds true in quite wide generality. Further, with the principal solution one can classify all solutions, of sufficient regularity, to the Beltrami equation. These facts are summarized in the following version of Stoilow's factorization theorem, for which proof we cite to [6, Thm. 20.5.2].

Theorem 2.2. *Suppose $\mu(z)$ is supported in the unit disk \mathbb{D} , $|\mu(z)| < 1$ a.e. and*

$$\int_{\mathbb{D}} \exp(\mathcal{A}(K_{\mu}(z))) dm(z) < \infty, \quad K_{\mu}(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}$$

where \mathcal{A} satisfies the almost linear growth condition (1.25). Then the equation

$$\bar{\partial}\Phi(z) = \mu(z) \partial\Phi(z), \quad z \in \mathbb{C}, \quad (2.17)$$

$$\Phi(z) = z + \mathcal{O}(1/z) \quad \text{as } z \rightarrow \infty \quad (2.18)$$

has a unique solution in $\Phi \in W_{loc}^{1,1}(\mathbb{C})$. The solution $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ is a homeomorphism and satisfies $\Phi \in W_{loc}^{1,P}(\mathbb{C})$. Moreover, when $\Omega_1 \subset \mathbb{C}$ is open, every solution of the equation

$$\bar{\partial}f(z) = \mu(z) \partial f(z), \quad z \in \Omega_1, \quad (2.19)$$

with the regularity $f \in W_{loc}^{1,P}(\Omega_1)$, can be written as $f = H \circ \Phi$, where Φ is the solution to (2.17)-(2.18) and H is holomorphic function in $\Omega'_1 = \Phi(\Omega_1)$.

Below we apply this results with the Poincare lemma to analyze the solutions of conductivity equation in the simply connected domain Ω .

Corollary 2.3. *Let $\sigma \in \Sigma_{\mathcal{A}}(\Omega)$ where \mathcal{A} satisfies (1.25) and $u \in W_{loc}^{1,1}(\Omega)$ satisfy*

$$\nabla \cdot \sigma \nabla w = 0 \quad \text{in } \Omega \quad \text{and} \quad \int_{\Omega} \nabla u(x) \cdot \sigma(x) \nabla u(x) dm(x) < \infty. \quad (2.20)$$

Then $u = w \circ \Phi$, where $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ is a homeomorphism, $\Phi \in W_{loc}^{1,P}(\mathbb{C})$, and w is harmonic in the domain $\Omega' = \Phi(\Omega)$. In particular, $u : \Omega \rightarrow \mathbb{R}$ is continuous.

Proof. Let $v \in W_{loc}^{1,1}(\Omega)$ be the conjugate function described in (2.5), and set $f = u + iv$. Then by Lemma 1.10 we have $f \in W^{1,P}(\Omega)$ and Theorem 2.2 yields that $f = H \circ \Phi$, where $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ a homeomorphism with $\Phi \in W_{loc}^{1,P}(\mathbb{C})$ and H is holomorphic in $\Phi(\Omega)$. Thus the real part $u = (\operatorname{Re} H) \circ \Phi$ has the required factorization. \square

Theorem 2.1 and Corollary 2.3 yield Theorem 1.4.

2.3. Invariance of Dirichlet-to-Neumann form in coordinate transformations. In this section, we assume that $\sigma \in \Sigma_{\mathcal{A}}(\Omega)$, where \mathcal{A} satisfies (1.25). We say that $F : \Omega \rightarrow \Omega'$ satisfies the condition \mathcal{N} if for any measurable set $E \subset \Omega$ we have $|E| = 0 \Rightarrow |F(E)| = 0$. Also, we say that F satisfies the condition \mathcal{N}^{-1} if for any measurable set $E \subset \Omega$ we have $|F(E)| = 0 \Rightarrow |E| = 0$.

Let $\sigma \in \Sigma_{\mathcal{A}}(\mathbb{C})$ be such that σ is constant 1 in $\mathbb{C} \setminus \Omega$. Let

$$\widehat{\mu}(z) = \frac{\sigma^{11}(z) - \sigma^{22}(z) + 2i\sigma^{12}(z)}{\sigma^{11}(z) + \sigma^{22}(z) + 2\sqrt{\det \sigma(z)}} \quad (2.21)$$

be the Beltrami coefficient associated to the isothermal coordinates corresponding to σ , see e.g. [59], [6, Thm. 10.1.1]. A direct computation shows that $K_{\widehat{\mu}}(z) = K_{\sigma}(z)$ and thus $\exp(\mathcal{A}(K_{\widehat{\mu}})) \in L_{loc}^1(\mathbb{C})$ and by Theorem 2.2, there exists a homeomorphism $F : \mathbb{C} \rightarrow \mathbb{C}$ satisfying the equation (2.17)-(2.18) with the Beltrami coefficient $\widehat{\mu}$ such that $F \in W_{loc}^{1,P}(\mathbb{C})$. Due to the choice of $\widehat{\mu}$, the conductivity $F_*\sigma$ is isotropic, see e.g. [59], [6, Thm. 10.1.1]. Let us next consider the properties of the map F . First, as $\exp(\mathcal{A}(K_{\widehat{\mu}})) \in L_{loc}^1(\mathbb{C})$, it follows from [33] that the function F satisfies the condition \mathcal{N} . Moreover, the fact that $K_F = K_{\widehat{\mu}} \in L_{loc}^1(\mathbb{C})$ implies by (2.16) that its inverse $H = F^{-1}$ is in $W_{loc}^{1,2}(\mathbb{C})$. This yields by [6, Thm. 3.3.7] that F^{-1} satisfies the condition \mathcal{N} . In particular, the above yields that both F and F^{-1} are in $W_{loc}^{1,P}(\mathbb{C})$.

The following lemma formulates the invariance of the Dirichlet-to-Neumann forms in the diffeomorphisms satisfying the above properties.

Lemma 2.4. *Assume that $\Omega, \widetilde{\Omega} \subset \mathbb{C}$ are bounded, simply connected domains with smooth boundaries and that $\sigma \in \Sigma_{\mathcal{A}}(\Omega)$ and $\widetilde{\sigma} \in \Sigma_{\mathcal{A}}(\widetilde{\Omega})$ where \mathcal{A} satisfies (1.25). Let $F : \Omega \rightarrow \widetilde{\Omega}$ be a homeomorphism so that F and F^{-1} are $W^{1,P}$ -smooth and F satisfies conditions \mathcal{N} and \mathcal{N}^{-1} . Assume that F and F^{-1} are C^1 smooth near the boundary and assume that $\rho = F|_{\partial\Omega}$ is C^2 -smooth. Also, suppose $\widetilde{\sigma} = F_*\sigma$. Then $Q_{\widetilde{\sigma}}[\widetilde{h}] = Q_{\sigma}[\widetilde{h} \circ \rho]$ for all $\widetilde{h} \in H^{1/2}(\partial\widetilde{\Omega})$.*

Proof. As F has the properties \mathcal{N} and \mathcal{N}^{-1} we have the area formula

$$\int_{\widetilde{\Omega}} H(y) dm(y) = \int_{\Omega} H(F(x)) J(x, F) dm(x) \quad (2.22)$$

for all simple functions $H : \tilde{\Omega} \rightarrow \mathbb{C}$, where $J(x, F)$ is the Jacobian determinant of F at x . Thus (2.22) holds for all $H \in L^1(\tilde{\Omega})$.

Let $\tilde{h} \in H^{1/2}(\partial\tilde{\Omega})$ and assume that $Q_{\tilde{\sigma}}[\tilde{h}] < \infty$. Let $\tilde{u} : \tilde{\Omega} \rightarrow \mathbb{R}$ be the unique minimizer of $A_{\tilde{\sigma}}[v]$ in $\tilde{X} = \{\tilde{v} \in W^{1,1}(\tilde{\Omega}); \tilde{v}|_{\partial\tilde{\Omega}} = \tilde{h}\}$. Then \tilde{u} is the solution of the conductivity equation

$$\nabla \cdot \tilde{\sigma} \nabla \tilde{u} = 0, \quad \tilde{u}|_{\partial\tilde{\Omega}} = \tilde{h}. \quad (2.23)$$

We define $h = \tilde{h} \circ F|_{\partial\Omega}$ and $u = \tilde{u} \circ F : \Omega \rightarrow \mathbb{C}$.

By Corollary 2.3, \tilde{u} can be written in the form $\tilde{u} = \tilde{w} \circ \tilde{G}$ where \tilde{w} is harmonic and $\tilde{G} \in W_{loc}^{1,1}(\mathbb{C})$ is a homeomorphism $\tilde{G} : \mathbb{C} \rightarrow \mathbb{C}$.

By Gehring-Lehto theorem, see [6, Cor. 3.3.3], a homeomorphism $F \in W_{loc}^{1,1}(\Omega)$ is differentiable almost everywhere in Ω , say in the set $\Omega \setminus A$, where A has Lebesgue measure zero. Similar arguments for \tilde{G} show that \tilde{G} and the solution \tilde{u} are differentiable almost everywhere, say in the set $\tilde{\Omega} \setminus A'$, where A' has Lebesgue measure zero.

Since F has the property \mathcal{N}^{-1} , we see that $A'' = A' \cup F^{-1}(A') \subset \Omega$ has measure zero, and for $x \in \Omega \setminus A''$ the chain rule gives

$$Du(x) = (D\tilde{u})(F(x)) \cdot DF(x). \quad (2.24)$$

Note that the facts that F is a map with an exponentially integrable distortion and that \tilde{u} is a real part of a map with an exponentially integrable distortion, do not generally imply, at least according to the knowledge of the authors, that their composition u is in $W_{loc}^{1,1}(\Omega)$. To overcome this problem, we define for $m > 1$

$$\tilde{\Omega}_m = \{y \in \tilde{\Omega}; \|DF^{-1}(y)\| + \|DF(F^{-1}(y))\| + \|\tilde{\sigma}(y)\| + |\nabla \tilde{u}(y)| < m\}$$

and $\Omega_m = F^{-1}(\tilde{\Omega}_m)$. Then $\nabla u|_{\Omega_m} \in L^2(\Omega_m)$ and $\|\sigma\| < m^3$ in Ω_m .

Now for any $m > 0$

$$\int_{\tilde{\Omega}_m} \nabla \tilde{u}(y) \cdot \tilde{\sigma}(y) \nabla \tilde{u}(y) dm(y) \leq A_{\tilde{\sigma}}[\tilde{u}] < \infty. \quad (2.25)$$

Due to the definition of $\tilde{\sigma} = F_*\sigma$, we see by using formulae (2.22) and (2.24) that

$$\int_{\Omega_m} \nabla u(x) \cdot \sigma(x) \nabla u(x) dm(x) = \int_{\tilde{\Omega}_m} \nabla \tilde{u}(y) \cdot \tilde{\sigma}(y) \nabla \tilde{u}(y) dm(y). \quad (2.26)$$

Letting $m \rightarrow \infty$ and using monotone convergence theorem, we see that

$$\int_{\Omega} \nabla u(x) \cdot \sigma(x) \nabla u(x) dm(x) = \int_{\tilde{\Omega}} \nabla \tilde{u}(y) \cdot \tilde{\sigma}(y) \nabla \tilde{u}(y) dm(y) = A_{\tilde{\sigma}}[\tilde{u}] < \infty. \quad (2.27)$$

By Lemma 1.10 this implies that $u \in W^{1,P}(\Omega) \subset W^{1,1}(\Omega)$.

Clearly, as $\rho = F|_{\partial\Omega}$ is C^2 -smooth $h := \tilde{h} \circ F \in H^{1/2}(\partial\Omega)$ and $u|_{\partial\Omega} = h$. Thus $u \in X = \{w \in W^{1,1}(\Omega); w|_{\partial\Omega} = h\}$. Since \tilde{u} is a minimizer of $A_{\tilde{\sigma}}$ in \tilde{X} , and u satisfies $A_{\sigma}[u] \leq A_{\tilde{\sigma}}[\tilde{u}] = Q_{\tilde{\sigma}}(\tilde{h})$ and see that

$$Q_{\sigma}[h] \leq Q_{\tilde{\sigma}}[\tilde{h}].$$

Changing roles of $\tilde{\sigma}$ and σ we obtain an opposite inequality, and prove the claim. \square

In particular, if $\sigma \in \Sigma_{\mathcal{A}}(\Omega)$, $\tilde{\sigma} \in \Sigma_{\mathcal{A}}(\tilde{\Omega})$ and F are as in Lemma 2.4 and in addition to that, σ and $\tilde{\sigma}$ are bounded near $\partial\Omega$ and $\partial\tilde{\Omega}$, respectively and $\rho = F|_{\partial\Omega} : \partial\Omega \rightarrow \partial\tilde{\Omega}$ is C^2 -smooth, then the quadratic forms Q_{σ} and $Q_{\tilde{\sigma}}$ can be written in terms of the Dirichlet-to-Neumann maps $\Lambda_{\sigma} : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$ and $\Lambda_{\tilde{\sigma}} : H^{1/2}(\partial\tilde{\Omega}) \rightarrow H^{-1/2}(\partial\tilde{\Omega})$ as in formula (1.4). Then, Lemma 2.4 implies that

$$\Lambda_{\tilde{\sigma}} = \rho_* \Lambda_{\sigma}, \quad (2.28)$$

where $\rho_* \Lambda_{\sigma}$ is the push forward of Λ_{σ} in ρ defined by $(\rho_* \Lambda_{\sigma})(\tilde{h}) = j \cdot [(\Lambda_{\sigma}(\tilde{h} \circ \rho)) \circ \rho^{-1}]$ for $\tilde{h} \in H^{1/2}(\partial\tilde{\Omega})$, where $j(z)$ is the Jacobian of the map $\rho^{-1} : \partial\tilde{\Omega} \rightarrow \partial\Omega$.

2.4. Counterexamples revisited. In this section we give the proofs of the claims stated in Subsection 1.2. We start by proving Theorem 1.6. Since the used singular change of variables in integration is a tricky business we present the arguments in detail.

Proof (of Thm. 1.6). (i) Our aim is first to show that we have $Q_{\sigma}[h] \leq Q_{\tilde{\sigma}}[h]$ and then to prove the opposite inequality. The proofs of these inequalities are based on different techniques due to the fact that $\tilde{\sigma}$ is not even in $L^1(B(2))$.

Let $0 < r < 2$ and $\mathcal{K}(r) = \mathcal{K} \cup F(\overline{B}(r))$. Moreover, let $\tilde{\sigma}_r$ be a conductivity that coincide with $\tilde{\sigma}$ in $B(2) \setminus \mathcal{K}(r)$ and is 0 in $\mathcal{K}(r)$. Similarly, let σ_r be a conductivity that coincide with σ in $B(2) \setminus \overline{B}(r)$ and is 0 in $\overline{B}(r)$. For these conductivities we define the quadratic forms $A^r : W^{1,1}(B(2)) \rightarrow \mathbb{R}_+ \cup \{0, \infty\}$ and $\tilde{A}^r : W^{1,1}(B(2)) \rightarrow \mathbb{R}_+ \cup \{0, \infty\}$,

$$A^r[v] = \int_{B(2) \setminus \overline{B}(r)} \nabla v \cdot \sigma \nabla v \, dm(x), \quad \tilde{A}^r[v] = \int_{B(2) \setminus \mathcal{K}(r)} \nabla v \cdot \tilde{\sigma} \nabla v \, dm(x).$$

If we minimize $\tilde{A}^r[v]$ over $v \in W^{1,1}(B(2))$ with $v|_{\partial B(2)} = h$, we see that minimizers exist and that the restriction of any minimizer to $B(2) \setminus \overline{\mathcal{K}}(r)$ is the function $\tilde{u}_r \in W^{1,2}(B(2) \setminus \mathcal{K}(r))$ satisfying

$$\nabla \cdot \tilde{\sigma} \nabla \tilde{u}_r = 0 \quad \text{in } B(2) \setminus \mathcal{K}(r), \quad \tilde{u}_r|_{\partial B(2)} = h, \quad \nu \cdot \tilde{\sigma} \nabla \tilde{u}_r|_{\partial \mathcal{K}(r)} = 0.$$

Analogous equations hold for the minimizer u^r of A^r . As σ in $\overline{B}(2) \setminus B(r)$ and $\tilde{\sigma}$ in $\overline{B(2)} \setminus \overline{\mathcal{K}(r)}$ are bounded from above and below by positive constants, we

using the change of variables and the chain rule that

$$Q_{\sigma_r}[h] = Q_{\tilde{\sigma}_r}[h], \quad \text{for } h \in H^{1/2}(\partial B(2)). \quad (2.29)$$

As $\sigma(x) \geq \sigma_r(x)$ and $\tilde{\sigma}(x) \geq \tilde{\sigma}_r(x)$ for all $x \in B(2)$,

$$Q_\sigma[h] \geq Q_{\sigma_r}[h], \quad Q_{\tilde{\sigma}}[h] \geq Q_{\tilde{\sigma}_r}[h]. \quad (2.30)$$

Let us consider the minimization problem (1.3) for σ . It is solved by the unique minimizer $u \in W^{1,1}(B(2))$ satisfying

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } B(2), \quad u|_{\partial B(2)} = h.$$

As $\sigma, \sigma^{-1} \in L^\infty(B(2))$ we have $u \in W^{1,2}(B(2))$ and Morrey's theorem [49] yields that the solution u is $C^{0,\alpha}$ -smooth in the open ball $B(2)$ for some $\alpha > 0$. Thus $u|_{B(R)}$ is in the Royden algebra $\mathcal{R}(B(R)) = C(B(R)) \cap L^\infty(B(R)) \cap W^{1,2}(B(R))$ for all $R < 2$.

By e.g. [31, p. 443], for any $0 < R < 2$ the p -capacity of the disc $B(r)$ in $B(R)$ goes to zero as $r \rightarrow 0$ for all $p > 1$. Using this, and that $u \in W^{1,2}(B(2)) \subset L^q(B(2))$ for $q < \infty$, we see that (cf. [36] for explicit estimates in the case when $\sigma = 1$)

$$\lim_{r \rightarrow 0} Q_{\sigma_r}[h] = Q_\sigma[h],$$

that is, the effect of an insulating disc of radius r in the boundary measurements vanishes as $r \rightarrow 0$. This and the inequalities (2.29) and (2.30) yield $Q_{\tilde{\sigma}}[h] \geq Q_\sigma[h]$. Next we consider the opposite inequality.

Let $\tilde{u} = u \circ F^{-1}$ in $B(2) \setminus \mathcal{K}$. As F is a homeomorphism, we see that if $x \rightarrow 0$ then $d(F(x), \mathcal{K}) \rightarrow 0$ and vice versa. Thus, as u is continuous at zero, we see that $\tilde{u} \in C(B(2) \setminus \mathcal{K}^{int})$ and \tilde{u} has the constant value $u(0)$ on $\partial \mathcal{K}$. Moreover, as $F^{-1} \in C^1(B(2) \setminus \mathcal{K})$, $\|DF^{-1}\| \leq C_0$ in $B(2) \setminus \mathcal{K}$ and u is in the Royden algebra $\mathcal{R}(B(R))$ for all $R < 2$, we have by [6, Thm. 3.8.2] that the chain rule holds implying that $D\tilde{u} = ((Du) \circ F^{-1}) \cdot DF^{-1}$ a.e. in $B(2) \setminus \mathcal{K}$. Let $0 < R' < R'' < 2$. Then

$$|D\tilde{u}(z)| \leq C_0 \|Du\|_{C(\overline{B(R'')})}, \quad \text{for } z \in F(B(R'')) \setminus \mathcal{K}.$$

As F and F^{-1} are C^1 smooth up to $\partial B(2)$, $\tilde{u} \in W^{1,1}(B(2) \setminus B(R'))$. These give $\tilde{u} \in W^{1,1}(B(2) \setminus \mathcal{K})$. Let $\tilde{v} \in W^{1,1}(B(2))$ be a function that coincides with \tilde{u} in $B(2) \setminus \mathcal{K}$ and with $u(0)$ in \mathcal{K} .

Again, using the chain rule and the area formula as in the proof of Lemma 2.4 we see that $\tilde{A}^r[\tilde{v}] = A^r[u]$ for $r > 1$. Applying monotone convergence theorem twice, we obtain

$$Q_{\tilde{\sigma}}[h] \leq A_{\tilde{\sigma}}[\tilde{v}] = \lim_{r \rightarrow 0} \tilde{A}^r[\tilde{v}] = \lim_{r \rightarrow 0} A^r[u] = Q_\sigma[h]. \quad (2.31)$$

As we have already proven the opposite inequality, this proves the claim (i).

(ii) Assume that $\tilde{\sigma}$ is a cloaking conductivity obtained by the transformation map F and background conductivity $\sigma \in L^\infty(B(2))$, $\sigma \geq c_1 > 0$ but that opposite

to the claim, we have $\text{tr}(\tilde{\sigma}) \in L^1(B(2) \setminus \mathcal{K})$. Using formula (1.6) and the facts that $\det(\tilde{\sigma}) = \det(\sigma \circ F^{-1})$ is bounded from above and below by strictly positive constants and $\text{tr}(\tilde{\sigma}) \in L^1(B(2) \setminus \mathcal{K})$, we see that $\text{tr}(\tilde{\sigma}^{-1}) \in L^1(B(2) \setminus \mathcal{K})$. Hence by Lemma 1.2, $K_{\tilde{\sigma}} \in L^1(B(2) \setminus \mathcal{K})$. Let $G : B(2) \setminus \mathcal{K} \rightarrow B(2) \setminus \{0\}$ be the inverse map of F . Using the formulas (1.5), (1.13), and (2.15) we see that

$$\|\tilde{\sigma}(y)\| = \frac{\|DF(x) \cdot \sigma(x) \cdot DF(x)^t\|}{J(x, F)} \geq \frac{\|DF(x)\|^2}{J(x, F)K_{\sigma}(x)} = \frac{K_F(x)}{K_{\sigma}(x)}, \quad x = F^{-1}(y).$$

As $K_G = K_F \circ F^{-1}$, cf. [6, formula (2.15)] and $\|\tilde{\sigma}(y)\| \leq K_{\tilde{\sigma}}(y)$, the above yields $K_G \in L^1(B(2) \setminus \mathcal{K})$. Hence, we see using (2.16) that $F = G^{-1}$ is in $W^{1,2}(B(2) \setminus \{0\})$ and $\|DF\|_{L^2(B(2) \setminus \{0\})} \leq 2\|K_G\|_{L^1(B(2) \setminus \mathcal{K})}$. By the removability of singularities in Sobolev spaces, see [35], this implies that $F : B(2) \setminus \{0\} \rightarrow B(2) \setminus \mathcal{K}$ can be extended to a function $F^{ext} : B(2) \rightarrow \mathbb{C}$, $F^{ext} \in W^{1,2}(B(2))$. As the distortion K_F of the map F is finite a.e., also the map F^{ext} is a finite distortion map, see [6, Def. 20.0.3]. Thus, as $F^{ext} \in W_{loc}^{1,2}(B(2))$, it follows from the continuity theorem of finite distortion maps [6, Thm. 20.1.1] or [48] that $F^{ext} : B(2) \rightarrow \mathbb{C}$ is continuous. Let $y_0 = F(0)$. Then the set $F^{ext}(\overline{B}(2)) = (\overline{B}(2) \setminus \mathcal{K}) \cup \{y_0\}$ is not closed as ∂K contains more than one point and thus it is not compact. This is a contradiction with the fact that F^{ext} is continuous. This proves the claim (ii). \square

Next we prove the claim concerning the last counterexample.

Proof of Theorem 1.7. Let us start by reviewing the properties of the Iwaniec-Martin maps. Let $\mathcal{A}_1 : [1, \infty] \rightarrow [0, \infty]$ be a strictly increasing positive smooth function with $\mathcal{A}_1(1) = 0$ which satisfies the condition (1.19). Then by [31, Thm. 11.2.1] there exists a $W^{1,1}$ -homeomorphism $F : B(2) \setminus \{0\} \rightarrow B(2) \setminus \overline{B}(1)$ which Beltrami coefficient μ satisfies

$$\int_{B(2) \setminus \{0\}} \exp(\mathcal{A}_1(K_{\mu}(z))) \, dm(z) < \infty, \quad \text{where} \quad K_{\mu}(z) := \frac{1 + |\mu(z)|}{1 - |\mu(z)|}. \quad (2.32)$$

The function F can be obtained using the construction procedure of [6, Sec. 20.3] (see [31, Thm. 11.2.1] the original construction) as follows: Let $S(t)$ be solution of the equation

$$\mathcal{A}_1(S(t)) = 1 + \log(t^{-1}), \quad 0 < t \leq 1. \quad (2.33)$$

Then $S : (0, 1] \rightarrow [1, \infty)$ is well defined decreasing function, $S(1) = 1$ and with suitably chosen $c_1 > 0$, the function

$$F(z) = \frac{z}{|z|} \rho(|z|), \quad \rho(s) = 1 + c_1 \left(\exp \left(\int_0^s \frac{dt}{t S(t)} \right) - 1 \right) \quad (2.34)$$

is a homeomorphism $F : B(2) \setminus \{0\} \rightarrow B(2) \setminus \overline{B}(1)$. We say that F is the Iwaniec-Martin map corresponding to the weight function $\mathcal{A}_1(t)$.

Next let $\mathcal{A} : [1, \infty] \rightarrow [0, \infty]$ be a strictly increasing positive smooth function with $\mathcal{A}(1) = 0$ which satisfies the condition (1.19) and let F_1 be the Iwaniec-Martin map corresponding to the weight function $\mathcal{A}_1(t) = \mathcal{A}(4t)$.

Using the inverse of the map F_1 we define $\sigma_1 = (F_1^{-1})_* 1$ on $B(2) \setminus \{0\}$ and consider this function as an a.e. defined measurable function on $B(2)$. Using the definition of push forward, (2.32), we see that $\det(\sigma_1) = 1$ and $K_{\sigma_1}(z) = K(F_1^{-1}(z), F_1^{-1}) = K_\mu(z)$. Thus Lemma 1.2 and the fact that F_1 satisfies (2.32) with the weight function $\mathcal{A}_1(t) = \mathcal{A}(4t)$ yield that σ_1 satisfies (1.20) with the weight function $\mathcal{A}(t)$.

Recall that the conductivity γ_1 that is identically 1 in $B(2) \setminus \overline{B}(1)$ and zero in $\overline{B}(1)$. Next, we consider the minimization problem (1.3) with the conductivities γ_1 and σ_1 . To this end, we make analogous definitions to the proof of Thm. 1.6. For $1 < r < 2$ let γ_r be a conductivity that is 1 in $B(2) \setminus B(r)$ and is 0 in $B(r)$. Similarly, let σ_r be a conductivity that coincide with σ_1 in $B(2) \setminus B(r-1)$ and is 0 in $B(r-1)$.

As in (2.29) and (2.30), we see for $h \in H^{1/2}(\partial B(2))$ and $r > 1$

$$Q_{\sigma_r}[h] = Q_{\gamma_r}[h], \quad Q_{\sigma_r}[h] \leq Q_{\sigma_1}[h], \quad Q_{\gamma_r}[h] \leq Q_{\gamma_1}[h]. \quad (2.35)$$

Let $h \in H^{1/2}(\partial B(2))$. For $1 \leq r < 2$ the solution of the boundary value problem

$$\Delta w_r = 0 \quad \text{in } B(2) \setminus \overline{B}(r), \quad w_r|_{\partial B(2)} = h, \quad \partial_\nu w_r|_{\partial B(r)} = 0$$

satisfies $Q_{\gamma_r}[h] = \|\nabla w_r\|_{L^2(B(2) \setminus \overline{B}(r))}^2$ and it is easy to see (c.f. [32]) that

$$\lim_{r \rightarrow 0} Q_{\gamma_r}[h] = Q_{\gamma_1}[h], \quad \text{for } h \in H^{1/2}(\partial B(2)). \quad (2.36)$$

Let $w = w_1$. Note that $w \in W^{1,2}(B(2) \setminus \overline{B}(1))$.

Let us consider the function $v = w \circ F_1$. As F_1 is C^1 -smooth in $\overline{B}(2) \setminus \{0\}$ and the function w is C^1 -smooth in $\overline{B}(R) \setminus \overline{B}(1)$ for all $1 < R < 2$ we have by the chain rule $Dv(x) = (Dw)(F_1(x)) \cdot DF_1(x)$ for all $x \in B(2) \setminus \{0\}$. As $Dw \in L^2(B(2) \setminus B(R))$ and $Dw \in L^\infty(B(R) \setminus \overline{B}(1))$ for all $1 < R < 2$, and

$$DF_1(x) = \frac{\rho(|x|)}{|x|}(I - P(x)) + \rho'(|x|)P(x)$$

where $P(x) : y \mapsto |x|^{-2}(x \cdot y)x$ is the projector to the radial direction at the point x , we using (2.34) that see that $\|DF_1(x)\| \leq C|x|^{-1}$ with some $C > 0$ and

$$Dv \in L^p(B(2) \setminus \{0\}), \quad \text{for any } p \in (1, 2). \quad (2.37)$$

Thus $v \in W^{1,p}(B(2) \setminus \{0\})$ with any $p \in (1, 2)$ and by the removability of singularities in Sobolev spaces, see e.g. [35], function v can be considered as a measurable function in $B(2)$ for which $v \in W^{1,p}(B(2))$. Thus v is in the domain of definition of the quadratic form A_{σ_1} .

As $w \in C^1(\overline{B}(R) \setminus \overline{B}(1))$ for all $1 < R < 2$ and F_1 is C^1 -smooth in $\overline{B}(2) \setminus \overline{B}(1)$, we can use again the chain rule, the area formula, and the monotone convergence theorem to obtain

$$\begin{aligned} Q_{\sigma_1}[h] &\leq A_{\sigma_1}[v] = \lim_{R \rightarrow 2} \lim_{\rho \rightarrow 0} \int_{B(R) \setminus \overline{B}(\rho)} \nabla v \cdot \sigma_1 \nabla v \, dm(x) \\ &= \lim_{R \rightarrow 2} \lim_{\rho \rightarrow 0} \int_{F_1(B(R) \setminus \overline{B}(\rho))} \nabla w \cdot \gamma_1 \nabla w \, dm(x) = Q_{\gamma_1}[h]. \end{aligned} \quad (2.38)$$

Next, consider an inequality opposite to (2.38). We have by (2.35) and (2.36)

$$Q_{\sigma_1}[h] \geq \lim_{r \rightarrow 1} Q_{\sigma_r}[h] = \lim_{r \rightarrow 1} Q_{\gamma_r}[h] = Q_{\gamma_1}[h]. \quad (2.39)$$

The above inequalities prove the claim. \square

3. COMPLEX GEOMETRIC OPTICS SOLUTIONS

In this section we assume that \mathcal{A} satisfies the almost linear growth condition (1.25).

3.1. Existence and properties of the complex geometric optics solutions.

Let us start with the observation that if $\sigma_0 \in \Sigma(\Omega_0)$ is a conductivity in a smooth simply connected domain $\Omega_0 \subset \mathbb{C}$, and σ_1 is a conductivity in a larger smooth domain Ω_1 which coincides with σ_0 in Ω_0 and is one in $\Omega_1 \setminus \Omega_0$, then Q_{σ_0} determines Q_{σ_1} by formula

$$Q_{\sigma_1}[h] = \inf \left\{ \int_{\Omega_1 \setminus \Omega_0} |\nabla v|^2 \, dm(z) + Q_{\sigma_0}[v|_{\partial\Omega_0}] ; v \in W^{1,2}(\Omega_1 \setminus \overline{\Omega_0}), v|_{\partial\Omega_1} = h \right\}.$$

This observation implies that we may consider inverse problems by assuming that the conductivity σ is the identity near $\partial\Omega$ without loss of generality. Also, we may assume that $\Omega = \mathbb{D}$, which we do below.

The main result of this section is the following uniqueness and existence theorem for the complex geometrical optics solutions.

Theorem 3.1. *Let $\sigma \in \Sigma_{\mathcal{A}}(\mathbb{C})$ be a conductivity such that $\sigma(x) = 1$ for $x \in \mathbb{C} \setminus \Omega$. Then for every $k \in \mathbb{C}$ there is a unique solution $u(\cdot, k) \in W_{loc}^{1,P}(\mathbb{C})$ for*

$$\begin{aligned} \nabla_z \cdot \sigma(z) \nabla_z u(z, k) &= 0, \quad \text{for a.e. } z \in \mathbb{C}, \\ u(z, k) &= e^{ikz} \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) \quad \text{as } |z| \rightarrow \infty. \end{aligned}$$

We point out that the regularity $u \in W_{loc}^{1,P}(\mathbb{C})$ is optimal in the sense that a slightly stronger assumption $u \in W_{loc}^{1,2}(\mathbb{C})$ would not be valid for the solutions and a slightly weaker assumption $u \in \bigcap_{1 < q < 2} W_{loc}^{1,q}(\mathbb{C})$ would not be strong enough for obtaining the uniqueness, see [6] and the equivalence of the conductivity equation and the Beltrami equation discussed in (2.4)-(2.7).

We prove Thm. 3.1 in several steps. Recalling the reduction to the Beltrami equation (2.7), we start with the following lemma, where we denote

$$B_{\mathcal{A}}(\mathbb{D}) = \{\mu \in L^\infty(\mathbb{C}) \ ; \ \text{supp}(\mu) \subset \overline{\mathbb{D}}, \ 0 \leq \mu(x) < 1 \ \text{a.e.}, \text{ and} \\ \int_{\mathbb{D}} \exp(\mathcal{A}(K_\mu(z))) \, dm(z) < \infty\}.$$

Lemma 3.2. *Assume that $\mu \in B_{\mathcal{A}}(\mathbb{D})$ and $f \in W_{loc}^{1,P}(\mathbb{C})$ satisfies*

$$\bar{\partial}f(z) = \mu(z)\partial f(z), \quad \text{for a.e. } z \in \mathbb{C}, \quad (3.1)$$

$$f(z) = \beta e^{ikz}(1 + \mathcal{O}(\frac{1}{z})) \quad \text{for } |z| \rightarrow \infty, \quad (3.2)$$

where $\beta \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{C}$. Then

$$f(z) = \beta e^{ik\Phi(z)}, \quad (3.3)$$

where $\Phi \in W_{loc}^{1,P}(\mathbb{C})$ is a homeomorphism $\Phi : \mathbb{C} \rightarrow \mathbb{C}$, $\bar{\partial}\Phi(z) = 0$ for $|z| > 1$, $K(z, \Phi) = K(z, f)$ for a.e. $z \in \mathbb{C}$, and

$$\Phi(z) = z + \mathcal{O}(\frac{1}{z}) \quad \text{for } |z| \rightarrow \infty. \quad (3.4)$$

Proof. By Theorem 2.2 we have for f the Stoilow factorization $f = h \circ \Phi$ where $h : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic function and Φ is the principal solution of (3.1). This and the formulae (3.2) and (3.4) imply

$$\frac{h(\Phi(z))}{\beta e^{ik\Phi(z)}} = \frac{f(z)}{\beta e^{ik\Phi(z)}} \rightarrow 1 \quad \text{when } |z| \rightarrow \infty.$$

Thus, $h(\zeta) = \beta e^{ik\zeta}$ for all $\zeta \in \mathbb{C}$, and f has the representation (3.3). The claimed properties of Φ follows from the formula (3.3) and the similar properties of f . \square

Next we consider case where $\beta = 1$. Below we will use the fact that if $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ is a homeomorphism such that $\Phi \in W_{loc}^{1,1}(\mathbb{C})$, $\Phi(z) - z = o(1)$ as $z \rightarrow \infty$ and that Φ is analytic outside the disc $\overline{B}(r)$, $r > 0$ then by [6, Thm. 2.10.1 and (2.61)]

$$|\Phi(z)| \leq |z| + 3r \text{ for } z \in \mathbb{C} \text{ and } |\Phi(z) - z| \leq r \text{ for } |z| > 2r. \quad (3.5)$$

In particular, the map Φ defined in Lemma 3.2 satisfies this with $r = 1$.

Lemma 3.3. *Assume that $\nu, \mu : \mathbb{C} \rightarrow \mathbb{C}$ are measurable functions satisfying*

$$\mu(z) = \nu(z) = 0, \quad \text{for } z \in \mathbb{C} \setminus \mathbb{D}, \quad (3.6)$$

$$|\mu(z)| + |\nu(z)| < 1, \quad \text{for a.e. } z \in \mathbb{D}, \quad (3.7)$$

and that $K_{\mu,\nu}(z)$ defined in (2.9) satisfies

$$\int_{\mathbb{D}} \exp(\mathcal{A}(K_{\mu,\nu}(z))) \, dm(z) < \infty. \quad (3.8)$$

Then for $k \in \mathbb{C}$ the equation

$$\partial_{\bar{z}}f = \mu\partial_zf + \nu\overline{\partial_zf}, \quad z \in \mathbb{C} \quad (3.9)$$

has at most one solution $f \in W_{loc}^{1,P}(\mathbb{C})$ satisfying

$$f(z) = e^{ikz}(1 + \mathcal{O}(\frac{1}{z})) \quad \text{for } |z| \rightarrow \infty. \quad (3.10)$$

Proof. Observe that we can write equation (3.9) in the form

$$\partial_{\bar{z}}f = \tilde{\mu}\partial_zf, \quad z \in \mathbb{C}, \quad (3.11)$$

where the coefficient $\tilde{\mu}$ is given by (2.13). Since $|\tilde{\mu}(z)| \leq |\mu(z)| + |\nu(z)|$, we see that $\tilde{\mu} \in B_{\mathcal{A}}(\mathbb{D})$.

Next, assume equation (3.11) has two solutions f_1 and f_2 having the asymptotics (3.10). Let $\varepsilon > 0$ and consider function $f_\varepsilon(z) = f_1(z) - (1 + \varepsilon)f_2(z)$. Then, $f_\varepsilon \in W_{loc}^{1,P}(\mathbb{C})$, function f_ε satisfies (3.9), and

$$f_\varepsilon(z) = -\varepsilon e^{ikz}(1 + \mathcal{O}(\frac{1}{z})) \quad \text{for } |z| \rightarrow \infty.$$

By Lemma 3.2 and (3.5), there is $\Phi_\varepsilon(z)$ such that

$$f_\varepsilon(z) = f_1(z) - (1 + \varepsilon)f_2(z) = -\varepsilon e^{ik\Phi_\varepsilon(z)}$$

and $|\Phi_\varepsilon(z)| \leq |z| + 3$. Then for any $z \in \mathbb{C}$ we have that

$$f_1(z) - f_2(z) = \lim_{\varepsilon \rightarrow 0} f_\varepsilon(z) = 0.$$

Thus $f_1 = f_2$. □

3.2. Proof of Theorem 3.1. In following, we use general facts for weakly monotone mappings, and for this end, we recall some basic facts. Let $\Omega \subset \mathbb{C}$ be open and $u \in W^{1,1}(\Omega)$ be real valued. We say that u is weakly monotone, if the both functions $u(x)$ and $-u(x)$ satisfy the maximum principle in the following weak sense: For any $a \in \mathbb{R}$ and relatively compact open sets $\Omega' \subset \Omega$,

$$\max(u(z) - a, 0) \in W_0^{1,1}(\Omega') \text{ implies that } u(z) \leq a \text{ for a.e. } z \in \Omega',$$

see [31, Sec. 7.3]. We remark that if $f \in W_{loc}^{1,1}(\Omega_1)$ and $f : \Omega_1 \rightarrow \Omega_2$ is homeomorphism, where $\Omega_1, \Omega_2 \subset \mathbb{C}$ are open, the real part of f is weakly monotone. By [6, Lem. 20.5.8], if $f \in W^{1,1}(\Omega)$ is the solution of the Beltrami equation $\bar{\partial}f = \mu\partial f$ with a Beltrami coefficient μ satisfying $|\mu(z)| < 1$ for a.e. $z \in \mathbb{C}$, then the real and the imaginary parts of f are weakly monotone functions. An important property of weakly monotone functions is that their modulus of continuity can be estimated in an explicit way. Let $M_P(t)$ be the P -modulus, that is, the function determined by the condition

$$\text{For } M = M(t) \text{ we have } \int_1^{1/t} P(sM) \frac{ds}{s^3} = P(1),$$

cf. (1.30). The function $M_P : [0, \infty) \rightarrow [0, \infty)$ is continuous at zero and $M_P(0) = 0$. Then by [31, Thm. 7.5.1] it holds that if $z', z \in \Omega$ satisfy $B(z, r) \subset \Omega$, $r < 1$ and $|z' - z| < r/2$, and $f \in W^{1,P}(\Omega)$ is a weakly monotone function, then for almost every $z, z' \in B(z, r)$ we have

$$|f(z') - f(z)| \leq 32\pi r \|\nabla f\|_{(P,r)} M_P\left(\frac{|z - z'|}{2r}\right), \quad (3.12)$$

where

$$\|\nabla f\|_{(P,r)} = \inf \left\{ \frac{1}{\lambda}; \lambda > 0, \frac{1}{\pi r^2} \int_{B(z,r)} P(\lambda |\nabla f(x)|) dm(z) \leq P(1) \right\}.$$

As we will see, this can be used to estimate the modulus of continuity of principal solutions of Beltrami equations corresponding to $\mu \in B_A(\mathbb{D})$.

Below, we use the unimodular function e_k given by $e_k(z) = e^{i(kz + \bar{k}\bar{z})}$. The following lemma shows the existence of the complex geometric solutions for degenerated conductivities.

Lemma 3.4. *Assume that μ and ν satisfy (3.6), (3.7), (3.8) and let $k \in \mathbb{C} \setminus \{0\}$. Then the equation (3.9) has a solution $f \in W_{loc}^{1,P}(\mathbb{C})$ satisfying asymptotics (3.10). Moreover, this solution can be written in the form*

$$f(z) = e^{ik\varphi(z)} \quad (3.13)$$

where $\varphi : \mathbb{C} \rightarrow \mathbb{C}$ is a homeomorphism satisfying the asymptotics $\varphi(z) = z + \mathcal{O}(z^{-1})$. Moreover, for $R > 1$

$$\int_{B(R)} P(|D\varphi(x)|) dm(x) \leq C_A(R) \int_{B(R)} \exp(\mathcal{A}(K_{\mu,\nu}(z))) dm(z), \quad (3.14)$$

where $C_A(R)$ depends on R and the weight function \mathcal{A} . In addition,

$$\bar{\partial}\varphi(z) = \mu(z)\partial\varphi(z) - \frac{\bar{k}}{k}\nu(z)e_{-k}(\varphi(z))\overline{\partial\varphi(z)}, \quad \text{for a.e. } z \in \mathbb{C}. \quad (3.15)$$

Proof. Let us approximate the functions μ and ν with functions

$$\mu_n(z) = \begin{cases} \mu(z) & \text{if } |\mu(z)| + |\nu(z)| \leq 1 - \frac{1}{n}, \\ \frac{\mu(z)}{|\mu(z)|}(1 - \frac{1}{n}) & \text{if } |\mu(z)| + |\nu(z)| > 1 - \frac{1}{n}, \end{cases} \quad (3.16)$$

$$\nu_n(z) = \begin{cases} \nu(z) & \text{if } |\mu(z)| + |\nu(z)| \leq 1 - \frac{1}{n}, \\ \frac{\nu(z)}{|\nu(z)|}(1 - \frac{1}{n}) & \text{if } |\mu(z)| + |\nu(z)| > 1 - \frac{1}{n}, \end{cases} \quad (3.17)$$

where $n \in \mathbb{Z}_+$. Consider the equations

$$\bar{\partial}f_n(z) = \mu_n(z)\partial f_n(z) + \nu_n(z)\overline{\partial f_n(z)}, \quad \text{for a.e. } z \in \mathbb{C}, \quad (3.18)$$

$$f_n(z) = e^{ikz}(1 + \mathcal{O}(\frac{1}{z})) \quad \text{for } |z| \rightarrow \infty. \quad (3.19)$$

By Lemma 3.3 equations (3.18)-(3.19) have at most one solution $f_n \in W_{loc}^{1,P}(\mathbb{C})$. The existence of the solutions can be seen as in the proof of [8, Lem. 3.5]: By

[8, Lem. 3.2], solutions f_n for (3.18)-(3.19) can be constructed via the formula $f_n = h \circ g$, where g is the principal solution of $\bar{\partial}g = \hat{\mu}\partial g$, constructed in Thm. 2.2 and h is the solution of $\bar{\partial}h = (\hat{\nu} \circ g^{-1})\bar{\partial}h$, $h(z) = e^{ikz}(1 + \mathcal{O}(\frac{1}{z}))$ constructed in [7, Thm. 4.2] where $\hat{\nu} = (1 + \nu_n)\bar{\mu}_n$ and $\hat{\mu} = \mu_n(1 + \hat{\nu})$ and moreover, it holds that $f_n \in W_{loc}^{1,2}(\mathbb{C})$.

Let us now define the coefficient $\tilde{\mu}$ according to formula (2.13), and define an approximative coefficient $\tilde{\mu}_n$ using formula (2.13) where μ and ν are replaced by μ_n and ν_n and f by f_n . We can write the equation (3.18) in the form

$$\bar{\partial}f_n(z) = \tilde{\mu}_n(z)\partial f_n(z), \quad \text{for a.e. } z \in \mathbb{C} \quad (3.20)$$

where $|\tilde{\mu}_n| \leq 1 - n^{-1}$.

By (3.19), (3.20), and Lemma 3.2, function f_n can be written in the form

$$f_n(z) = e^{ik\varphi_n(z)}, \quad (3.21)$$

where φ_n is a homeomorphism, $\bar{\partial}\varphi_n(z) = 0$ for $|z| > 1$, $K(z, \varphi_n) = K(z, f_n)$ for a.e. $z \in \mathbb{C}$, and

$$\varphi_n(z) = z + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{for } |z| \rightarrow \infty. \quad (3.22)$$

Then

$$|\bar{\partial}f_n(z)| = |\tilde{\mu}_n(z)| |\partial f_n(z)| \leq |\tilde{\mu}(z)| |\partial f_n(z)|.$$

Let us consider next $a, b > 0$ and $0 \leq t \leq (ab)^{1/2}$. Using the definition (1.26) of $P(t)$ we see

$$\begin{aligned} P(t) &\leq \exp(\mathcal{A}(a)), & \text{for } t^2 \leq e^{\mathcal{A}(a)}, \\ P(t) &\leq \frac{ab}{\mathcal{A}^{-1}(\log \exp(\mathcal{A}(a)))} = b, & \text{for } t^2 > e^{\mathcal{A}(a)}, \end{aligned}$$

which imply the inequality $P(t) \leq b + \exp(\mathcal{A}(a))$. Due to the distortion equality (2.15) we can use this for $a = K(z, \varphi_n)$, $b = J(z, \varphi_n)$, and $t = |D\varphi_n(z)|$ and obtain

$$P(|D\varphi_n(z)|) \leq J(z, \varphi_n) + \exp(\mathcal{A}(K(z, \varphi_n))). \quad (3.23)$$

Then, we see using (3.5) and the fact that φ_n is a homeomorphism that

$$\begin{aligned} \int_{B(R)} P(|D\varphi_n(z)|) dm(z) &\leq \int_{B(R)} J(z, \varphi_n) dm(z) + \int_{B(R)} e^{\mathcal{A}(K(z, \varphi_n))} dm(z) \\ &\leq m(\varphi_n(B(R))) + \int_{B(R)} \exp(\mathcal{A}(K_{\tilde{\mu}}(z))) dm(z) \\ &\leq \pi(R+3)^2 + \int_{B(R)} \exp(\mathcal{A}(K_{\tilde{\mu}}(z))) dm(z) \end{aligned} \quad (3.24)$$

is finite by the assumption (3.8). We emphasize that the fact that φ_n is a homeomorphism is the essential fact which together with the inequality (3.23) yields the Orlicz estimate (3.24).

The estimate (3.24) together with the inequality (3.12) implies that the functions φ_n have uniformly bounded modulus of continuity in all compact sets of \mathbb{C} . Moreover, by (3.5), $|\varphi_n(z)| \leq |z| + 3$.

Next we consider the Beltrami equation for φ . To this end, let $\psi \in C_0^\infty(\mathbb{C})$ and $R > 1$ be so large that $\text{supp}(\psi) \subset B(R)$. Since the family $\{\varphi_n\}_{n=1}^\infty$ is uniformly bounded in the space $W^{1,P}(B(R))$ and $W^{1,P}(B(R)) \subset W^{1,q}(B(R))$ for some $q > 1$, we see that there is a subsequence φ_{n_j} that converges weakly in $W^{1,q}(B(R))$ to some limit φ when $j \rightarrow \infty$. Let us denote

$$\kappa_n(z) = -\frac{\bar{k}}{k}\nu_n(z)e_{-k}(\varphi_n(z)), \quad \kappa(z) = -\frac{\bar{k}}{k}\nu(z)e_{-k}(\varphi(z)).$$

Moreover, functions φ_n are uniformly bounded and have a uniformly bounded modulus of continuity in compact sets by (3.12) and thus by Arzela-Ascoli theorem there is a subsequence φ_{n_j} , denoted also by φ_{n_j} , that converges uniformly to some function φ' in $B(R)$ for all $R > 1$. As φ_{n_j} converge in $C(\overline{B}(R))$ uniformly to φ' and weakly in $W^{1,q}(B(R))$ to φ we see using convergence in distributions that $\varphi' = \varphi$. Thus, we see that

$$\lim_{j \rightarrow \infty} e_{-k}(\varphi_{n_j}(z)) = e_{-k}(\varphi(z)) \quad \text{uniformly for } z \in B(R),$$

and by dominated convergence theorem $\kappa_n \rightarrow \kappa$ in $L^p(B(R))$ where $1/p + 1/q = 1$.

As $\varphi_n : \mathbb{C} \rightarrow \mathbb{C}$ is a homeomorphism and $\varphi_n \in W_{loc}^{1,1}(\mathbb{C})$, we can use chain rules (2.10) a.e. by the Gehring-Lehto theorem, see [6, Cor. 3.3.3], and see using the equations (3.18) and (3.21) that

$$\bar{\partial}\varphi_n(z) = \mu_n(z)\partial\varphi_n(z) - \frac{\bar{k}}{k}\nu_n(z)e_{-k}(\varphi_n(z))\overline{\partial\varphi_n(z)}, \quad \text{for a.e. } z \in \mathbb{C}. \quad (3.25)$$

Recall that there is convex function $\Phi : [0, \infty) \rightarrow [0, \infty)$ such that $\Phi(t) \leq P(t) + c_0 \leq 2\Phi(t)$. By [10, Thm. 13.1.2] the map $\phi \mapsto \int_{B(R)} \Phi(|D\phi(x)|) dm(x)$ is weakly lower semicontinuous in $W^{1,1}(B(R))$. By (3.24) the integral of $\Phi(|D\varphi_n|)$ is uniformly bounded in $n \in \mathbb{Z}_+$ over any disc $B(R)$. In particular, this yields that $\varphi \in W^{1,P}(B(R))$ for $R > 1$ and that (3.14) holds.

Furthermore, as $|\varphi(z)| \leq |z| + 3$, this yields that

$$f(z) := e^{ik\varphi(z)} \in W_{loc}^{1,P}(\mathbb{C}). \quad (3.26)$$

Define next $\varphi_n(\infty) = \varphi(\infty) = \infty$. As φ_n and φ are conformal at infinity, we see using Cauchy formula for $(\varphi_n(1/z) - \varphi(0))^{-1}$ that that

$$\varphi(z) = z + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{for } |z| \rightarrow \infty. \quad (3.27)$$

As $D\varphi_{n_j}$ converges weakly in $L^q(B(R))$ to $D\varphi$ and their norms are uniformly bounded, we have

$$\begin{aligned} & \left| \int_{\mathbb{C}} (\bar{\partial}\varphi - \mu\partial\varphi - \kappa\bar{\partial}\varphi)\psi \, dm(z) \right| = \lim_{j \rightarrow \infty} \left| \int_{\mathbb{C}} (\bar{\partial}\varphi_{n_j} - \mu\partial\varphi_{n_j} - \kappa\bar{\partial}\varphi_{n_j})\psi \, dm(z) \right| \\ & \leq \lim_{j \rightarrow \infty} \left| \int_{\mathbb{C}} ((\mu_{n_j} - \mu)\partial\varphi_{n_j} + (\kappa_{n_j} - \kappa)\bar{\partial}\varphi_{n_j})\psi \, dm(z) \right| \\ & \leq \lim_{j \rightarrow \infty} (\|\mu_{n_j} - \mu\|_{L^p(B(1))} + \|\kappa_{n_j} - \kappa\|_{L^p(B(1))}) \|\partial\varphi_{n_j}\|_{L^q(B(1))} \|\psi\|_{L^\infty(B(1))} = 0. \end{aligned}$$

This implies that $\varphi(z)$ satisfies (3.15).

Next we show that φ is homeomorphism. As $K(z) = K_{\nu,\mu} \in L^1_{loc}(\mathbb{C})$, we have $K(z; \varphi_n) \in L^1_{loc}(\mathbb{C})$ thus by (2.16) the inverse maps φ_n^{-1} satisfy $\varphi_n^{-1} \in W^{1,2}_{loc}(\mathbb{C}; \mathbb{C})$ and for all $R > 1$ the norms $\|\varphi_n^{-1}\|_{W^{1,2}(B(R))}$, $n \in \mathbb{Z}_+$ are uniformly bounded. Thus by the formula (3.12) the family $(\varphi_n^{-1})_{n=1}^\infty$ has a uniform modulus on continuity in compact sets. Hence, we see that there is a continuous function $\psi : \mathbb{C} \rightarrow \mathbb{C}$ such that $\varphi_n^{-1} \rightarrow \psi$ uniformly on compact sets when $n \rightarrow \infty$. As φ_n are conformal at infinity, we see using again the Cauchy formula that $\varphi_{n_j}^{-1} \rightarrow \psi$ uniformly on the Riemann sphere \mathbb{S}^2 as $j \rightarrow \infty$. Then,

$$\psi \circ \varphi(z) = \lim_{j \rightarrow \infty} \varphi_{n_j}^{-1}(\varphi(z)) = \lim_{j \rightarrow \infty} \varphi_{n_j}^{-1}(\varphi_{n_j}(z)) = z$$

which implies that $\varphi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is a continuous injective map and hence a homeomorphism.

As $\varphi : \mathbb{C} \rightarrow \mathbb{C}$ is a homeomorphism and $\varphi \in W^{1,1}_{loc}(\mathbb{C})$, we can by the Gehring-Lehto theorem use chain rules (2.10) a.e. and see using the equation (3.15) that $f(z) = e^{ik\varphi(z)}$ satisfies equation (3.9). By (3.27), $f(z)$ satisfies asymptotics (3.10). This proves the claim. \square

The above uniqueness and existence results have now proven Theorem 3.1. \square

4. INVERSE CONDUCTIVITY PROBLEM WITH DEGENERATE ISOTROPIC CONDUCTIVITY

In this section we consider exponentially integrable scalar conductivities σ . In particular, we assume that σ is 1 in an open set containing $\mathbb{C} \setminus \mathbb{D}$ and its ellipticity function $K(z) = K_\sigma(z)$ of the conductivity σ satisfies an Orlicz space estimates

$$\int_{\mathcal{N}(R_1)} \exp(\exp(qK(x))) \, dm(x) \leq C_0 \quad \text{for some } C_0, q > 0 \quad (4.1)$$

with $R_1 = 1$. Note that by John-Nirenberg lemma, (4.1) is satisfied if

$$\exp(qK(x)) \in BMO(\mathbb{D}), \quad \text{for some } q > 0. \quad (4.2)$$

As noted before, we may assume without loss of generality that Ω is the unit disc \mathbb{D} .

4.1. Estimates for principal solutions in Orlicz spaces. Let us consider next the principal solution of the Beltrami equation

$$\bar{\partial}\Phi(z) = \mu(z) \partial\Phi(z), \quad z \in \mathbb{C} \quad (4.3)$$

$$\Phi(z) = z + O\left(\frac{1}{z}\right) \quad \text{when } |z| \rightarrow \infty. \quad (4.4)$$

For this end, let $R_0 \geq 1$ and

$$B_{exp,N}^p(B(R_0)) = \{\mu : \mathbb{C} \rightarrow \mathbb{C}; |\mu(z)| < 1 \text{ for a.e. } z, \text{ supp }(\mu) \subset B(R_0),$$

$$\text{and } \int_{B(R_0)} \exp(pK_\mu(z)) dm(z) \leq N\}$$

and $B_{exp}^p(B(R_0)) = \bigcup_{N>0} B_{exp,N}^p(B(R_0))$. The reason that we use the radius R_0 is to be able to apply the obtained results for the inverse function of the solution of the Beltrami equation satisfying another Beltrami equation with modified coefficients, see (4.45).

Assume that $p > 2$ and $\mu \in B_{exp}^p(B(R_0))$. Then by [5, Thm. 1.1] we have the L^2 -estimate

$$\|(\mu S)^m \mu\|_{L^2(\mathbb{C})} \leq C(p, \beta) m^{-\beta/2} \int_{B(R_0)} \exp(pK_\mu(z)) dm(z), \quad 2 < \beta < p. \quad (4.5)$$

In particular, as Φ satisfies

$$\bar{\partial}\Phi = \bar{\partial}(\Phi - z) = \mu \partial(\Phi - z) + \mu = \mu S \bar{\partial}(\Phi - z) + \mu = \mu S \bar{\partial}\Phi + \mu,$$

where

$$S\phi(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial_w \phi(w)}{w - z} dm(w),$$

is the Beurling operator, (4.5) yields

$$\bar{\partial}\Phi = \sum_{m=0}^{\infty} (\mu S)^m \mu, \quad (4.6)$$

where the series converges in $L^2(\mathbb{C})$. To analyze the convergence more precisely, we need a refinement of the L^p -scale. In particular, we will use the Orlicz spaces $X^{j,q}(S)$, $j \in \mathbb{Z}_+$, $q \in \mathbb{R}$, $S \subset \mathbb{C}$ that are defined by

$$u \in X^{j,q}(S) \quad \text{if and only if} \quad \int_S M_{j,q}(u(x)) dm(x) < \infty \quad (4.7)$$

where

$$M_{j,q}(t) = |t|^j \log^q(e + |t|). \quad (4.8)$$

We use shorthand notations $X^q(S) = X^{2,q}(S)$ and $M_q(t) = M_{2,q}(t)$. Although (4.7)-(4.8) do not define a norm in $X^{j,q}(S)$, there is an equivalent norm

$$\|u\|_{X^{j,q}(S)} = \sup \left\{ \int_S |u(x)v(x)| dm(x) ; \int_D G_{j,q}(|u(x)|) dm(x) \leq 1 \right\}. \quad (4.9)$$

where $G_{j,q}(t)$ is such function that $(M_{j,q}, G_{j,q})$ are a Young complementary pair (c.f. the Appendix) and, in particular, the following lemma holds.

Lemma 4.1. *We have for $j = 1, 2, \dots, q \geq 0$*

- (i) $\int_{B(R_0)} M_{j,q}(u(x)) dm(x) \leq 2 \|u\|_{X^{j,q}(B(R_0))}^j \log^q(e + \|u\|_{X^{j,q}(B(R_0))}),$
- (ii) $\|u\|_{X^{j,q}(B(R_0))} \leq \phi(\int_{B(R_0)} M_{j,q}(u(x)) dm(x)), \phi(t) = t^{1/j}(1 + 2 \log^q(e + t^{-1/j})).$

Proof. (i). Let us denote $M(t) = M_{j,q}(t)$. For this function we use the equivalent norms $\|u\|_M$ and $\|u\|_{(M)}$ defined in the Appendix. To show the claim we use the inequality

$$\log(e + st) \leq 2 \log(e + s) \log(e + t), \quad t, s \geq 0. \quad (4.10)$$

Let us consider function $w \in X^{j,q}(B(R_0))$. By (4.10) we have for $k > 0$

$$\begin{aligned} \int_{B(R_0)} M_{j,q}(kw) dm &= k^j \int_{B(R_0)} |w|^j \log^q(e + k|w|) dm \\ &\leq 2k^j \log^q(e + k) \int_{B(R_0)} M_{j,q}(w) dm. \end{aligned} \quad (4.11)$$

A function $u \in X^{j,q}(B(R_0))$ can be written as $u = kw$ where $k = \|u\|_{(M)}$ and $\|w\|_{(M)} = 1$. Then by (5.16)-(5.17) we have $\int_{B(R_0)} M_{j,q}(w) dm = 1$ hence (4.11) and (5.15) yield the claim (i).

(ii) Using (4.11) and the definition (5.13) of the Orlicz norm, we see that for all $k > 0$

$$\|u\|_{X^{j,q}(B(R_0))} \leq \frac{1}{k} (1 + \int_{B(R_0)} M_{j,q}(ku) dm) \leq \frac{1}{k} (1 + 2k^j \log^q(e + k) \int_{B(R_0)} M_{j,q}(u) dm).$$

Let $T = \int_{B(R_0)} M_{j,q}(u) dm$. Substituting above $k = T^{-1/j}$ we obtain (ii). \square

Theorem 4.2. *Assume that $\mu \in B_{exp}^p(B(R_0))$, $2 < p < \infty$. Then the equations (4.3)-(4.4) have a unique solution $\Phi \in W_{loc}^{1,1}(\mathbb{C})$ which satisfies for $0 \leq q \leq p/4$*

$$\bar{\partial}\Phi \in X^q(\mathbb{C}) \quad (4.12)$$

and the series (4.6) converges in $X^q(\mathbb{C})$. The convergence of the series (4.6) in $X^q(\mathbb{C})$ is uniform for $\mu \in B_{exp,N}^p(B(R_0))$ with any $N > 0$. Moreover, for $\mu \in B_{exp,N}^p(B(R_0))$ the Jacobian $J_\Phi(z)$ of Φ satisfies

$$\|J_\Phi\|_{X^{1,q}(B(R))} \leq C \quad (4.13)$$

where C depends on p, q, N , and R . Moreover, let $s > 2$ and assume that $\mu_m, \tilde{\mu}_m \in B_{exp,N}^p(B(R_0))$ and $0 \leq q \leq p/4$. Then it holds that

$$\lim_{m \rightarrow \infty} \|\mu_m - \tilde{\mu}_m\|_{L^s(B(R_0))} = 0 \quad \Rightarrow \quad \lim_{m \rightarrow \infty} \|\bar{\partial}\Phi_m - \bar{\partial}\tilde{\Phi}_m\|_{X^q(\mathbb{C})} = 0 \quad (4.14)$$

where Φ_m and $\tilde{\Phi}_m$ are the solutions of (4.3)-(4.4) corresponding to $\mu_m, \tilde{\mu}_m$, respectively.

Proof. Let $\Phi^\alpha(z)$, $|\lambda| \leq 1$, $z \in \mathbb{C}$ be the principal solution corresponding to the Beltrami coefficient $\lambda\mu$, that is, solution with the Beltrami equation (2.17)-(2.18) with coefficient $\lambda\mu$. These solutions, in particular $\Phi^\lambda = \Phi^1$, exist and are unique by Thm. 2.2. It follows from [5, Thm. 1.1 and 5.1] that the Jacobian determinant $J\Phi^\alpha(z)$ of Φ^λ satisfies

$$\int_{B(R_0)} J\Phi^\lambda \log^{2q}(e + J\Phi^\lambda) dm(z) \leq C < \infty, \quad (4.15)$$

where C is independent of λ and $\mu \in B_{exp,N}^p(B(R_0))$ and depends only on N, p , and q . Thus (4.13) follows from Lemma 4.1 ii.

We showed already that when $p > 2$, $\bar{\partial}\Phi \in L^2(\mathbb{C})$ and that the series (4.6) converges in $L^2(\mathbb{C})$. To show the convergence of (4.6) in $X^q(\mathbb{C})$ and to prove (4.14), we present few lemmas in terms of Orlicz spaces $X^q(B(R_0))$ and the function M_q defined in (4.8). Note that as μ vanishes in $\mathbb{C} \setminus B(R_0)$, $\|(\mu S)^n \mu\|_{X^q(\mathbb{C})} = \|(\mu S)^n \mu\|_{X^q(B(R_0))}$.

Lemma 4.3. *Let $N \in \mathbb{Z}_+$, $2 < 2q < \beta < p$, and $\mu \in B_{exp,N}^p(B(R_0))$. Then*

$$\int_{B(R_0)} M_q(\psi_n(x)) dm(x) \leq cn^{-(\beta-q)} < cn^{-q}, \quad (4.16)$$

where $\psi_n = (\mu S)^n \mu$ and $c > 0$ depends only on N, p, β , and q .

Proof. Let $E_n = \{z \in B(R_0); |\psi_n(z)| \geq A^n\}$, where $A > 1$ is a constant to be chosen later. By (4.5),

$$\|\psi_n\|_{L^2(B(R_0))} \leq C_{N,\beta,p} n^{-\beta/2}. \quad (4.17)$$

Thus

$$|E_n| \leq C_{N,\beta,p}^2 A^{-2n} n^{-\beta}. \quad (4.18)$$

Using (4.17) we obtain

$$\int_{B(R_0) \setminus E_n} |\psi_n|^2 \log^q(e + |\psi_n|) dm \leq \|\psi_n\|_{L^2(B(R_0))}^2 \log^q(e + A^n) \leq C_1 n^{-\beta+q} \quad (4.19)$$

where $C_1 = C_{N,\beta,p}^2 \log^q(e + A)$.

The principal solution corresponding to the Beltrami coefficient $\lambda\mu$ can be written in the form $\Phi^\lambda = (I - \lambda\mu S)^{-1}(\lambda\mu)$. Expanding $\bar{\partial}_z \Phi^\alpha(z)$ as a power series in λ , we see that by (4.6) we can write using any ρ , $0 < \rho < 1$

$$\chi_{E_n}(z) \psi_n(z) = \frac{1}{2\pi i} \int_{|\lambda|=\rho} \lambda^{-n-2} \chi_{E_n}(z) \bar{\partial}_z \Phi^\alpha(z) d\lambda.$$

This gives

$$\|\chi_{E_n} \psi_n\|_{X^q(B(R_0))} \leq \rho^{-(n+2)} \sup_{|\lambda|=\rho} \|\chi_{E_n} \bar{\partial}_z \Phi^\lambda\|_{X^q(B(R_0))}. \quad (4.20)$$

Using the facts that $|\lambda| = \rho$ and that the Beltrami coefficient Φ^λ is bounded by $|\lambda|$, we have, by the distortion equality (2.15), $|\bar{\partial}_z \Phi^\alpha(z)|^2 \leq \rho^2(1 - \rho^2)^{-1} J\Phi^\alpha(z)$. Hence,

$$I := \int_{E_n} M_q(\bar{\partial}_z \Phi^\alpha(z)) dm(z) \leq \frac{\rho^2}{1 - \rho^2} \int_{E_n} J\Phi^\alpha \log^q(e + \left(\frac{\rho^2}{1 - \rho^2} J\Phi^\alpha\right)^{\frac{1}{2}}) dm.$$

Let \widehat{C} denote next a generic constant which is function of N, β, p, q and ρ but not of A . The above implies by (4.10), (4.15), and the inequality $\log(e + t^{1/2}) \leq 1 + \log(e + t)$, $t \geq 0$,

$$\begin{aligned} I &\leq \widehat{C} \int_{E_n} J\Phi^\alpha(z) (1 + \log(e + J\Phi^\alpha))^q dm(z) \\ &\leq \widehat{C} \left(\int_{E_n} J\Phi^\alpha(z) dm(z) \right)^{1/2} \left(\int_{E_n} J\Phi^\alpha(z) (1 + \log(e + J\Phi^\alpha))^{2q} dm(z) \right)^{1/2} \\ &\leq \widehat{C} \left(\int_{E_n} J\Phi^\alpha(z) dm(z) \right)^{1/2}. \end{aligned}$$

By the area distortion theorem [4],

$$\int_{E_n} J\Phi^\alpha(z) dm(z) \leq |\Phi^\alpha(E_n)| \leq \widehat{C} |E_n|^{1/M} \leq \widehat{C} A^{-2n/M},$$

where $M = (1 + \rho)/(1 - \rho) > 1$, and thus $I \leq \widehat{C} A^{-n/M}$. By Lemma 4.1 (ii) also $\|\chi_{E_n} \bar{\partial} \Phi^\lambda\|_{X^q} \leq \widehat{C} A^{-n/M}$. We take $\rho > e^{-1/2}$ and $A = e^M$ we see using (4.20) and Lemma 4.1 again that also $\int_{E_n} M_q(\psi_n) dm(z) \leq \widehat{C} e^{-n/2}$ as $n \rightarrow \infty$. Thus the assertion follows from (4.19). \square

Lemmas 4.1 (ii) and 4.3 and the fact that μ vanishes outside $B(R_0)$ yield that for any $N > 0$ the series (4.6) converges in $X^q(\mathbb{C})$, and the convergence of the series (4.6) in $X^q(\mathbb{C})$ is uniform for $\mu \in B_{exp,N}^p(B(R_0))$ where $q > 1$ and $p > 2q$. Thus to prove Theorem 4.2 it remains to show (4.14).

Lemma 4.4. *Let $2 < 2q < p$, $N > 0$, $2 < \beta < p$, $s > 2$, $\mu, \nu \in B_{exp,N}^p(B(R_0))$, and $B_n = (\mu S)^n \mu - (\nu S)^n \nu$. Then*

$$\sup_{n \in \mathbb{Z}_+} \int_{\mathbb{C}} M_q(B_n(x)) dm(x) \leq C \quad (4.21)$$

where $C > 0$ depends only on N, p , and q . Moreover, there is $T > 1$ such that

$$\|B_n\|_{L^2(\mathbb{C})} \leq C_{N,\beta,p,s,T} \min(nT^n \|\mu - \nu\|_{L^s(B(R_0))}, n^{-\beta/2}). \quad (4.22)$$

Proof. Lemmas 4.1 and 4.3 yield (4.21). Next, let us observe that for $z \in \mathbb{C}$

$$B_n(z) = (\mu S)^n \mu - (\nu S)^n \nu = \sum_{j=0}^n A_j(z), \quad A_j(z) = (\mu S)^j (\mu - \nu) (S\nu)^{n-j} \chi_{B(R_0)}.$$

As $\|\nu\|_{L^\infty} \leq 1$ and $\|S\|_q := \|S\|_{L^q(\mathbb{C}) \rightarrow L^q(\mathbb{C})} < \infty$ for $1 < q < \infty$, we have that

$$\begin{aligned} \int_{\mathbb{C}} |A_j(z)|^q dm(z) &\leq (\|S\|_q^q)^j \int_{B(R_0)} |\mu(z) - \nu(z)|^q |((S\nu)^{n-j} \chi_{B(R_0)})(z)|^q dm(z) \\ &\leq \|S\|_q^{jq} \left(\int_{B(R_0)} |\mu(z) - \nu(z)|^{q\rho} dm(z) \right)^{\frac{1}{\rho}} \left(\int_{B(R_0)} |((S\nu)^{n-j} \chi_{B(R_0)})(z)|^{q\rho'} dm(z) \right)^{\frac{1}{\rho'}} \end{aligned}$$

where $\rho^{-1} + (\rho')^{-1} = 1$, $1 < \rho < \infty$. Thus

$$\|A_j(z)\|_{L^q(\mathbb{C})} \leq (\|S\|_q^q)^j \|\mu - \nu\|_{L^{\rho q}(B(R_0))} (\|S\|_{q\rho'}^q)^{n-j} \|\nu\|_{L^{q\rho'}(B(R_0))}^q,$$

where $\|\nu\|_{L^{q\rho'}(B(R_0))} \leq \pi R_0^2$. Thus by choosing $q = 2$ and ρ so that $s = q\rho > 2$ yielding $q\rho' = 2s/(s-2)$, we obtain

$$\|(\mu S)^n \mu - (\nu S)^n \nu\|_{L^2(\mathbb{C})} \leq (n+1) \pi^2 R_0^4 (1 + \|S\|_{2s/(s-2)}^2)^n \|\mu - \nu\|_{L^s(B(R_0))}.$$

This and (4.5) show that (4.22) is valid. \square

Now we are ready to prove (4.14) which finishes the proof of Theorem 4.2. Let $B_{n,m} = (\mu_m S)^n \mu_m - (\tilde{\mu}_m S)^n \tilde{\mu}_m$. By Schwartz inequality, (4.21), (4.22) and Lemma 4.1 yield

$$\begin{aligned} \int_{B(R_0)} M_q(B_{n,m}(z)) dm(z) &\leq \int_{B(R_0)} |B_{n,m}|^2 \log^q(e + |B_{n,m}|) dm(z) \quad (4.23) \\ &\leq \left(\int_{B(R_0)} M_{2q}(B_{n,m}(z)) dm(z) \right)^{1/2} \|B_{n,m}\|_{L^2(B(R_0))} \\ &\leq C \min(nT^n \|\mu_m - \tilde{\mu}_m\|_{L^s(B(R_0))}, n^{-\beta/2}) \end{aligned}$$

where C depends only on q, p, β, s, T , and N .

Let $\varepsilon > 0$. As μ_m and $\tilde{\mu}_m$ vanish outside $B(R_0)$,

$$\|\bar{\partial}\Phi_m - \bar{\partial}\tilde{\Phi}_m\|_{X^q(\mathbb{C})} = \|\bar{\partial}\Phi_m - \bar{\partial}\tilde{\Phi}_m\|_{X^q(B(R_0))} \leq \sum_{n=0}^{\infty} \|B_{n,m}\|_{X^q(B(R_0))}.$$

Thus by (4.23) and Lemma 4.1 (ii) we can take $n_0 \in \mathbb{N}$ so large that for all m

$$\sum_{n=n_0}^{\infty} \|B_{n,m}\|_{X^q(B(R_0))} \leq \frac{\varepsilon}{2}.$$

Applying again (4.23) and Lemma 4.1 (ii) we can choose $\delta > 0$ so that

$$\sum_{n=0}^{n_0-1} \|B_{n,m}\|_{X^q(B(R_0))} \leq \frac{\varepsilon}{2} \quad \text{when } \|\mu_m - \tilde{\mu}_m\|_{L^s(B(R_0))} \leq \delta.$$

This proves Theorem 4.2. \square

Lemma 4.5. *Assume that K_μ corresponding to μ supported in \mathbb{D} satisfies (4.1) with $q, C_0 > 0$ and $R_1 = 1$. Let Φ is the principal solution of Beltrami equation corresponding to μ . Then for all $\beta, R > 0$ the inverse function $\Psi = \Phi^{-1} : \mathbb{C} \rightarrow \mathbb{C}$ of Φ satisfies*

$$\int_{B(R)} \exp(\beta K_\mu(\Psi(z))) dm(z) < C$$

where C depends on q, C_0, β , and R .

Proof. Since Φ satisfies the condition \mathcal{N} by [5, Cor. 4.3] we may change variable in integration to see that

$$\int_{B(R)} \exp(\beta K_\mu(\Psi(z))) dm(z) = \int_{\Psi(B(R))} \exp(\beta K_\mu(w)) J_\Phi(w) dm(w). \quad (4.24)$$

Using (3.5) for function Φ and $R > 3$ we see that $\Psi(B(R)) \subset \tilde{B} = B(\tilde{R})$, $\tilde{R} = R + 1$. By (4.1), $\exp(K_\mu(z)) \in L^q(\tilde{B})$ for all $q > 1$ and thus by (4.13) $J_\Phi \in X^{1,q}(B(R))$ for $R > 0$.

Let us next use properties of Orlicz spaces and the notations discussed in the Appendix using a Young complementary pair (F, G) where $F(t) = \exp(t^{1/p}) - 1$ and $G(t)$ satisfies $G(t) = C_p t (\log(1 + C_p t))^p$ for $t > T_p$ with suitable $C_p, T_p > 0$, see [39, Thm. I.6.1].

By using $u(z) = \exp(\beta K_\mu(z))$ and $v = J_\Phi(z)$ we obtain from Young's inequality (5.18) the inequality

$$\begin{aligned} & \int_{B(R)} \exp(\beta K_\mu(\Psi(z))) dm(z) \\ & \leq \int_{\tilde{B}} F(\exp(\beta K_\mu(w))) dm(w) + \int_{\tilde{B}} G(J_\Phi(w)) dm(w) \\ & \leq \int_{\tilde{B}} \exp((\exp(\beta K_\mu(w)))^{1/p}) dm(w) + \int_{\tilde{B}} C_p J_\Phi(w) (\log(1 + C_p J_\Phi(w)))^p dm(w). \end{aligned} \quad (4.25)$$

We apply this by using $p > \beta/q$, so that $(\exp(\beta K_\mu(w)))^{1/p} \leq \exp(q K_\mu(w))$. Thus

$$\int_{\tilde{B}} \exp((\exp(\beta K_\mu(w)))^{1/p}) dm(w) \leq \int_{\tilde{B}} \exp(\exp(q K_\mu(w))) dm(w) < \infty.$$

The last term in (4.25) is finite by (4.13), and thus the claim follows. \square

4.2. Asymptotics of the phase function of the exponentially growing solution. Let $\mu \in B_{exp}^p(B(R_0))$, $k \in \mathbb{C} \setminus \{0\}$ and $\lambda \in \mathbb{C}$ satisfy $|\lambda| \leq 1$. Then using Lemmas 3.3 and 3.4, with the affine weight $\mathcal{A}(t) = pt - p$ corresponding to

the gauge function Q , we see that the equation

$$\bar{\partial}_z f_k(z) = \lambda \frac{\bar{k}}{k} e_{-k}(z) \mu(z) \overline{\partial_z f_k(z)}, \quad \text{for a.e. } z \in \mathbb{C}, \quad (4.26)$$

$$f_k(z) = e^{ikz} (1 + O(\frac{1}{z})), \quad \text{as } |z| \rightarrow \infty \quad (4.27)$$

has the unique solution $f_k \in W_{loc}^{1,Q}(\mathbb{C})$. Moreover, this solution can be written in the form

$$f_k(z) = e^{ik\varphi_k(z)} \quad (4.28)$$

where $\varphi_k : \mathbb{C} \rightarrow \mathbb{C}$ is a homeomorphism satisfying

$$\bar{\partial}\varphi_k(z) = -\frac{\lambda \bar{k}}{k} \mu(z) e_{-k}(\varphi_k(z)) \overline{\partial\varphi_k(z)}, \quad \text{for a.e. } z \in \mathbb{C}, \quad (4.29)$$

$$\varphi_k(z) = z + \mathcal{O}(\frac{1}{z}), \quad \text{as } |z| \rightarrow \infty. \quad (4.30)$$

We denote below $f_k(z) = f(z, k)$ and $\varphi_k(z) = \varphi(z, k)$ and estimate next functions φ_k in the Orlicz space $X^q(\mathbb{C})$. The following lemma is a generalization of results of [7] to Orlicz space setting.

Lemma 4.6. *Assume that $\nu \in B_{exp}^p(B(R_0))$ for all $0 < p < \infty$. For $k \in \mathbb{C} \setminus \{0\}$ let $\Phi_k \in W^{1,1}(\mathbb{C})$ be the solution of*

$$\bar{\partial}\Phi_k(z) = -\frac{\bar{k}}{k} \nu(z) e_{-k}(z) \partial\Phi_k(z), \quad \text{for a.e. } z \in \mathbb{C}, \quad (4.31)$$

$$\Phi_k(z) = z + \mathcal{O}(\frac{1}{z}). \quad (4.32)$$

Then for all $\varepsilon > 0$ there exist $C_0 > 0$ such that $\bar{\partial}_z \Phi_k(z) = g_k(z) + h_k(z)$ where $g_k, h_k \in X^q(\mathbb{C})$ are supported in $B(R_0)$ and

$$\sup_{k \in \mathbb{C} \setminus \{0\}} \|h_k\|_{X^q} < \varepsilon, \quad (4.33)$$

$$\sup_{k \in \mathbb{C} \setminus \{0\}} \|g_k\|_{X^q} < C_0, \quad (4.34)$$

$$\lim_{k \rightarrow \infty} \widehat{g}_k(\xi) = 0, \quad (4.35)$$

where for all compact sets $S \subset \mathbb{C}$ the convergence in (4.35) is uniform for $\xi \in S$.

Proof. Let us denote $\tilde{\nu}_k(z) = \bar{k}k^{-1}\nu(z)$ for $k \in \mathbb{C} \setminus \{0\}$. Note that then for any $p > 0$ there is $N > 0$ such that $\tilde{\nu}_k(\cdot, k)e_{-k}(\cdot) \in B_{exp,N}^p(B(R_0))$ for all $k \in \mathbb{C} \setminus \{0\}$. By Theorem 4.2,

$$\lim_{n \rightarrow \infty} \|\bar{\partial}\Phi_k - \sum_{n=0}^{\infty} (\tilde{\nu}_k e_{-k} S)^n (\tilde{\nu}_k e_{-k})\|_{X^q(\mathbb{C})} = 0$$

uniformly in $k \in \mathbb{C} \setminus \{0\}$. We define

$$g_k(z) = \sum_{n=0}^m (\tilde{\nu}_k e_{-k} S)^n (\tilde{\nu}_k e_{-k}), \quad h_k(z) = \sum_{n=m+1}^{\infty} (\tilde{\nu}_k e_{-k} S)^n (\tilde{\nu}_k e_{-k})$$

For given $\varepsilon > 0$ we can choose m so large that (4.33) holds for all $k \in \mathbb{C} \setminus \{0\}$ and then using Lemma 4.3 choose C_0 so that (4.34) holds for all $k \in \mathbb{C} \setminus \{0\}$.

Next, we show (4.35) when ε and m fixed so that (4.33) and (4.34) hold. We can write

$$g_k(z) = \sum_{n=0}^m e_{-nk} G_n, \quad G_n = \left(\frac{\bar{k}}{k}\right)^{n+1} \nu S_n(k) \nu \dots \nu S_1(k) \nu$$

where $S_j(k)$ is the Fourier-multiplier

$$(S_j(k)\phi)^\wedge(\xi) = m(\xi + jk)\hat{\phi}(\xi), \quad m(\xi) = \frac{\bar{\xi}}{\xi}.$$

The proof of [7, Lemma 7.3] for $n \geq 1$ and the Riemann-Lebesgue lemma for $n = 0$ yields that for any $\tilde{\varepsilon} > 0$ there exists $R(n, \tilde{\varepsilon}) \geq 0$ such that

$$|\hat{G}_n(\xi)| \leq (n+1)\kappa^n \tilde{\varepsilon}, \quad \text{for } |\xi| > R(n, \tilde{\varepsilon}),$$

where $\kappa = \|\nu\|_{L^\infty} \leq 1$. Thus

$$|\hat{G}_n(\xi)| \leq (m+1)\tilde{\varepsilon}, \quad \text{for } |\xi| > R_0 = \max_{n \leq m} R(n, \tilde{\varepsilon}), \quad n = 0, 1, 2, \dots, m. \quad (4.36)$$

As $(e_{-nk} G_n)^\wedge(\xi) = \hat{G}_n(\xi - nk)$, we see that for any $L > 0$ there is $k_0 > 0$ such that if $|k| > k_0$ then $j|k| - L > R_0$ for $1 \leq n \leq m$. Then it follows from (4.36) that if $|k| > k_0$, then

$$\sup_{|\xi| < L} |\hat{g}_k(\xi)| \leq (m+1)^2 \tilde{\varepsilon}.$$

This proves the limit (4.35), with the convergence being uniform for ξ belonging in a compact set. \square

Proposition 4.7. *Assume that $\nu \in B_{exp}^p(B(R_0))$ with $p > 4$ and $\Phi_k(z)$ be the solution of (4.31)-(4.32). Then*

$$\lim_{k \rightarrow \infty} \Phi_k(z) = z \quad \text{uniformly for } z \in \mathbb{C}. \quad (4.37)$$

Proof. Step 1. We will first show that for all q with $4 < q < p$ we have $\bar{\partial}_z \Phi_k(z) \rightarrow 0$ weakly in $X^q(\mathbb{C})$ as $k \rightarrow \infty$. Let $\eta \in X^{-q}(\mathbb{C})$ and $\varepsilon_1 > 0$. By Theorem 4.2, there is $C_1 > 0$ such that $\sup_k \|\bar{\partial} \Phi_k\|_{X^q} \leq C_1$. Since $C_0^\infty(\mathbb{C})$ is dense in $X^{-q}(\mathbb{C})$, cf. [39, Sec. II.10], we can find a function $\eta_0 \in C_0^\infty(\mathbb{C})$ such that $\|\eta - \eta_0\|_{X^{-q}} \leq \min(1, \varepsilon_1/C_1)$. Then

$$|\langle \eta, \bar{\partial} \Phi_k \rangle| \leq |\langle \eta_0, \bar{\partial} \Phi_k \rangle| + \|\eta - \eta_0\|_{X^{-q}(\mathbb{C})} \|\bar{\partial} \Phi_k\|_{X^q(\mathbb{C})}, \quad (4.38)$$

where the second term on the right hand side is smaller than ε_1 . Moreover, by Lemma 4.6, we can write $\bar{\partial}\Phi_k = h_k + g_k$ so that (4.33)-(4.35) are satisfied for $\varepsilon = \varepsilon_1(\|\eta\|_{X^{-q}} + 1)^{-1}$ and some $C_0 > 0$. Then $|\langle \eta_0, h_k \rangle| \leq \varepsilon_1$.

Since $\widehat{\eta}_0$ is a rapidly decreasing function, $\widehat{g}_k(\xi)$ is uniformly bounded for $\xi \in \mathbb{C}$ and $k \in \mathbb{C} \setminus \{0\}$ by Lemma 4.6, and $\widehat{g}_k \rightarrow 0$ uniformly in all bounded domains as $k \rightarrow \infty$, we see that

$$\langle \eta_0, g_k \rangle = \langle \widehat{\eta}_0, \widehat{g}_k \rangle \rightarrow 0 \quad \text{as } k \rightarrow \infty \quad (4.39)$$

Combining these, we see that $\langle \eta_0, \bar{\partial}\Phi_k \rangle \rightarrow 0$ as $k \rightarrow \infty$, and thus $\bar{\partial}_z\Phi_k(z) \rightarrow 0$ weakly in $X^q(\mathbb{C})$ as $k \rightarrow \infty$.

Step 2. Next we show the pointwise convergence

$$\lim_{k \rightarrow \infty} \bar{\partial}_z\Phi_k(z) = 0. \quad (4.40)$$

For this end, we observe that the function $\eta_z(w) = \pi^{-1}(w-z)^{-1}\chi_{B(R_0)}(w)$ satisfies $\eta_z \in X^{-q}(\mathbb{C})$ for $q > 1$. Since $\Phi_k(z) - z = \mathcal{O}(\frac{1}{z})$ and $\bar{\partial}\Phi_k$ is supported in $\overline{B(R_0)}$, we have

$$\Phi_k(z) = z - \frac{1}{\pi} \int_{B(R_0)} (w-z)^{-1} \bar{\partial}_w\Phi_k(w) dm(w) = z - \langle \eta_z, \bar{\partial}\Phi_k \rangle. \quad (4.41)$$

As $\bar{\partial}\Phi_k \rightarrow 0$ weakly in $X^q(\mathbb{C})$, we see (4.40) holds for all $z \in \mathbb{C}$.

Step 3. By (3.12) and (3.14) we see that the family $\{\Phi_k(z)\}_{k \in \mathbb{C} \setminus \{0\}}$ of homeomorphisms has a uniform modulus of continuity in compact sets. Moreover, since

$$\sup_k \|\bar{\partial}\Phi_k\|_{L^1(\mathbb{C})} \leq \sup_k \|\bar{\partial}\Phi_k\|_{X^q(B(R_0))} = C_2 < \infty$$

we obtain by (4.40) for $|z| > R_0 + 1$

$$|\Phi_k(z) - z| = |\langle \eta_z, \bar{\partial}\Phi_k \rangle| \leq \frac{C}{|z|} \|\bar{\partial}\Phi_k\|_{L^1(\mathbb{C})} \leq \frac{CC_2}{|z|}. \quad (4.42)$$

Thus, as the functions $\{\Phi_k(z)\}_{k \in \mathbb{C} \setminus \{0\}}$ are uniformly equicontinuous in compact sets, (4.42) and the pointwise convergence (4.40) yield the uniform convergence (4.37). \square

4.3. Properties of the solutions of the non-linear Beltrami equation.

Let $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$ and $\mu(z)$ by supported in $B(R_0)$, $R_0 \geq 1$, and assume that $K = K_\mu$ satisfies (4.1) with $q, C_0 > 0$ and $R_1 = 1$, c.f. (2.14). Motivated by Lemma 3.4, we consider next the solutions φ_k of the equation

$$\bar{\partial}_z\varphi_\lambda(z, k) = -\lambda \frac{\bar{k}}{k} \mu(z) e_{-k}(\varphi_\lambda(z, k)) \overline{\partial_z\varphi_\lambda(z, k)}, \quad z \in \mathbb{C}, \quad (4.43)$$

$$\varphi_\lambda(z, k) = z + O\left(\frac{1}{z}\right) \quad \text{as } |z| \rightarrow \infty. \quad (4.44)$$

Let $\psi_\lambda(\cdot, k) = \varphi_\lambda(\cdot, k)^{-1}$ be the inverse function of $\varphi_\lambda(\cdot, k)$. A simple computation based on differentiation of the identity $\psi_\lambda(\varphi_\lambda(z, k), k) = z$ in the z variable shows that then

$$\bar{\partial}_z \psi_\lambda(z, k) = -\lambda \frac{\bar{k}}{k} \mu(\psi_\lambda(z, k)) e_{-k}(z) \partial_z \psi_\lambda(z, k), \quad z \in \mathbb{C}, \quad (4.45)$$

$$\psi_\lambda(z, k) = z + O\left(\frac{1}{z}\right), \quad \text{as } |z| \rightarrow \infty. \quad (4.46)$$

We consider the equations (4.43) and (4.45) simultaneously by defining the sets

$$B_\mu = \{(\varphi, \nu); |\nu| \leq |\mu| \text{ and } \varphi : \mathbb{C} \rightarrow \mathbb{C} \text{ is a homeomorphism} \\ \text{with } \bar{\partial}\varphi = \nu \bar{\partial}\varphi, \varphi(z) = z + O(z^{-1})\}$$

and

$$\mathcal{G}_\mu = \{g \in W_{loc}^{1,Q}(\mathbb{C}); \bar{\partial}g = (\nu \circ \varphi^{-1})\partial g, g(z) = z + O(z^{-1}), (\varphi, \nu) \in B_\mu\}.$$

Now $\exp(\exp(qK_\mu)) \in L^1(B(R_0))$ with some $0 < q < \infty$ and $|\nu| \leq |\mu|$ a.e. Then $K_\nu(z) \leq K_\mu(z)$ a.e. Let $\bar{\partial}\varphi = \nu \bar{\partial}\varphi$ in \mathbb{C} , $\varphi(z) = z + O(z^{-1})$, so that $\bar{\partial}\varphi = \tilde{\nu}\partial\varphi$ with $|\tilde{\nu}(z)| = |\nu(z)|$. Then for $\psi = \varphi^{-1}$ we have $K(z, \psi) = K_\nu(\psi(z))$. Thus by Lemma 4.5 we have

$$\sup_{g \in \mathcal{G}_\mu} \|\exp(\beta K(\cdot, g))\|_{L^1(B(R))} = \sup_{(\varphi, \nu) \in B_\mu} \|\exp(\beta K_\nu \circ \varphi^{-1})\|_{L^1(B(R))} < \infty \quad (4.47)$$

for all $\beta > 0$ and $R > 0$. Using this and Theorem 2.2, we see that the functions $g \in \mathcal{G}_\mu$ are homeomorphisms. Moreover, for $g \in \mathcal{G}_\mu$ the condition $g \in W_{loc}^{1,Q}(\mathbb{C})$ implies that $Dg \in X_{loc}^{-1}(\mathbb{C})$. Furthermore by (4.13), we have

$$\sup_{(\varphi, \nu) \in B_\mu} \|J_\varphi\|_{X^{1,q}(B(R))} < \infty \quad (4.48)$$

for all $q > 0$.

Lemma 4.8. *The set \mathcal{G}_μ is relatively compact in the topology of uniform convergence.*

Proof. Let $(\varphi, \nu) \in B_\mu$ and $\psi = \varphi^{-1}$ and $\bar{\partial}\tilde{g} = (\nu \circ \varphi^{-1})\partial g$, $g(z) = z + O(z^{-1})$.

As μ is supported in $B(R_0)$, the function φ is analytic outside $\overline{B}(R_0)$ we see using (3.5) for function φ that for $R > 0$ we have $\varphi(B(R)) \subset B(R + 3R_0)$, $\psi(B(R)) \subset B(R + 3R_0)$, and that ψ is analytic outside $\overline{B}(4R_0)$.

Thus (3.5) and the same arguments which we used to prove the estimate (3.24) yield that for $R > 0$

$$\begin{aligned} \|Q(|Dg|)\|_{L^1(B(R))} &\leq \pi(R + 3R_0)^2 + \int_{B(R)} \exp(qK_\nu(\psi(w)) - q) dm(w) \\ &\leq \pi(R + 3R_0)^2 + \int_{B(R+3R_0)} \exp(qK_\nu(z) - q) J_\varphi(z) dm(z), \end{aligned} \quad (4.49)$$

where $Q(t) = |t|^2 / \log(|t| + 1)$. We will next use Young's inequality (5.18) with the admissible pair (F, G) where (c.f. [39, Ch. 1.3])

$$F(t) = e^t - t - 1, \quad G(t) = (1 + t) \log(1 + t) - t. \quad (4.50)$$

By Young's inequality, we have

$$\begin{aligned} & \int_{B(R+3R_0)} \exp(qK_\nu(z) - q) J_\varphi(z) dm(z) \\ & \leq \int_{B(R+3R_0)} \exp(\exp(qK_\nu(w) - q)) dm(w) + \int_{B(R+3R_0)} (1 + J_\varphi(w)) \log(1 + J_\varphi(w)) dm. \end{aligned}$$

This, (4.1), and (4.49) show that there is a constant $C(R, \mu)$ such that for $g \in \mathcal{G}_\mu$

$$\|Q(|Dg|)\|_{L^1(B(R))} \leq C(R, \mu). \quad (4.51)$$

As $g \in \mathcal{G}_\mu$ are homeomorphism, this and (3.12) imply that functions $g \in \mathcal{G}_\mu$ are equicontinuous in compact sets of \mathbb{C} . As $\text{supp}(\nu \circ \psi) \subset B(4R_0)$, g is analytic outside the disc $B(4R_0)$ and the inequality (3.5) yields for $R > 0$ and $g \in \mathcal{G}_\mu$,

$$g(B(R)) \subset B(R + 12R_0).$$

These imply by Arzela-Ascoli theorem that the set $\{g|_{B(R)}; g \in \mathcal{G}_\mu\}$ is relatively compact in the topology of uniform convergence for any $R > 0$. Thus by using a diagonalization argument we see that for arbitrary sequence $g_n \in \mathcal{G}_\mu$, $n = 1, 2, \dots$ there is a subsequence g_{n_j} which converges uniformly in all discs $B(R)$, $R > 0$. Finally, by Young's inequality we get using the same notations as in (4.41) for $|z| > 4R_0 + 1$

$$\begin{aligned} |g_k(z) - z| &= \left| \frac{1}{\pi} \int_{B(4R_0)} (w - z)^{-1} \bar{\partial}_w g_k(w) dm(w) \right| \\ &\leq \frac{1}{\pi(|z| - 1)} \int_{B(4R_0)} (Q(|\bar{\partial}_w g_k(w)|) + G_0(1)) dm(w) \end{aligned} \quad (4.52)$$

where $Q(t)$ and $G_0(t) = |t|^2 \log(|t| + 1)$ form a Young complementary pair (c.f. Appendix). Thus $|g_k(z) - z| \leq C_\mu / (|z| - 1)$ for $|z| > 4R_0 + 1$. Using this and the uniform convergence of g_{n_j} in all discs $B(R)$, $R > 0$, we see that g_n has a subsequence converging uniformly in \mathbb{C} . \square

Theorem 4.9. *Let $\lambda, k \in \mathbb{C} \setminus \{0\}$, $|\lambda| = 1$. Assume that $\varphi_\lambda(z, k)$ satisfies (4.43)-(4.44) with μ supported in \mathbb{D} which satisfies (4.1) with $q > 0$ and $R_1 = 1$. Then*

$$\lim_{k \rightarrow \infty} \varphi_\lambda(z, k) = z$$

uniformly in $z \in \mathbb{C}$ and $|\lambda| = 1$.

Proof. Let $\psi_\lambda(\cdot, k)$ be the inverse function of $\varphi_\lambda(\cdot, k)$. It is sufficient to show that

$$\lim_{k \rightarrow \infty} \psi_\lambda(z, k) = z$$

uniformly in $z \in \mathbb{C}$ and $|\lambda| = 1$.

Then, $\psi_\lambda(\cdot, k)$ is the solution of (4.45)-(4.46). Denote $\nu(z) = -\lambda \bar{k} k^{-1} \mu(z)$ and note that $|\nu(z)| = |\mu(z)|$. Hence $(\varphi_\lambda(\cdot, k), \nu(\cdot) e_{-k}(\cdot)) \in B_\mu$ and $\psi_\lambda(\cdot, k) \in \mathcal{G}_\mu$. Moreover, as $\varphi_\lambda(\cdot, k)$ is homeomorphism in \mathbb{C} and analytic outside in $B(1)$, it follows from (3.5) that $\varphi_\lambda(\cdot, k)$ maps the ball $B(1)$ in to $B(3)$ and moreover, its inverse $\psi_\lambda(\cdot, k)$ maps the ball $B(3)$ in to $B(4)$ and $\mathbb{C} \setminus B(3)$ in to $\mathbb{C} \setminus B(2)$.

It follows from Lemma 4.8 that if the claim is not valid, there are sequences $(\lambda_n)_{n=1}^\infty$, $|\lambda_n| = 1$ and $(k_n)_{n=1}^\infty$, $k_n \rightarrow \infty$ such that

$$\psi_\infty(z) = \lim_{n \rightarrow \infty} \psi_{\lambda_n}(z, k_n),$$

where the convergence is uniform $z \in \mathbb{C}$, such that $\psi_\infty(z)$ is not equal to z . Thus, to prove the claim, it is enough to show that such a limit satisfies $\psi_\infty(z) = z$. Note that by considering subsequences, we can assume that $\lambda_n \rightarrow \lambda$ and $\bar{k}_n k_n^{-1} \rightarrow \beta$ as $n \rightarrow \infty$ where $|\lambda| = |\beta| = 1$. Denote next $\nu_0(z) = -\lambda \beta \mu(z)$

Let us consider the solution of

$$\bar{\partial}_z \Phi_\lambda(z, k) = \nu_0(\psi_\infty(z)) e_{-k}(z) \partial_z \Phi_\lambda(z, k), \quad (4.53)$$

$$\Phi_\lambda(z, k) = z + O\left(\frac{1}{z}\right) \quad \text{as } |z| \rightarrow \infty. \quad (4.54)$$

By Proposition 4.7, $\Phi_\lambda(z, k) \rightarrow z$ as $k \rightarrow \infty$ uniformly in $z \in \mathbb{C}$. Since for every $z \in \mathbb{C}$ the function $\eta_z : w \mapsto \chi_{B(4)}(w)(z - w)^{-1}$ is in $X^{-q}(\mathbb{C})$ for $q > 1$, we obtain using (4.41)

$$\begin{aligned} |\psi_{\lambda_n}(z, k_n) - \Phi_\lambda(z, k_n)| &= \frac{1}{\pi} \left| \int_{B(4)} (w - z)^{-1} \bar{\partial}_w (\psi_{\lambda_n}(w, k_n) - \Phi_\lambda(w, k_n)) dm(w) \right| \\ &\leq |\eta_z|_{X^{-q}(B(4))} \|\bar{\partial}(\psi_{\lambda_n}(\cdot, k_n) - \Phi_\lambda(\cdot, k_n))\|_{X^q(B(4))}. \end{aligned} \quad (4.55)$$

Let us next assume that we can prove that

$$\lim_{n \rightarrow \infty} \|\mu \circ \psi_{\lambda_n}(\cdot, k_n) - \mu \circ \psi_\infty(\cdot, k_n)\|_{L^s(\mathbb{C})} = 0, \quad \text{for some } s > 2. \quad (4.56)$$

If this is the case, let $p \in (4q, \infty)$. By assumption (4.1) and Lemma 4.5 there is N such that the Beltrami coefficients of functions $\psi_{\lambda_n}(\cdot, k_n)$ are in $B_{exp, N}^p(\mathbb{D})$ for all $n \in \mathbb{Z}_+$ and $p > 4$. By Theorem 4.2 and (4.56),

$$\lim_{n \rightarrow \infty} \|\bar{\partial}(\psi_{\lambda_n}(\cdot, k_n) - \Phi_\lambda(\cdot, k_n))\|_{X^q(\mathbb{C})} = 0.$$

As $\lim_{n \rightarrow \infty} \Phi_\lambda(z, k_n) = z$ uniformly in $z \in \mathbb{C}$, this and (4.55) shows that $\psi_\infty(z) = z$.

Thus, to prove the claim it is enough to show (4.56). First, as $\psi_{\lambda_n}(\cdot, k_n) \rightarrow \psi_\infty(\cdot)$ uniformly as $n \rightarrow \infty$ and as $\psi_{\lambda_n}(\cdot, k_n)$ maps $\mathbb{C} \setminus B(3)$ in $\mathbb{C} \setminus B(2)$, we see using Dominated convergence theorem that the formula (4.56) is valid when μ is replaced by a smooth compactly supported function. Next, let (F, G) be the complementary Young pair given by (4.50) and $E_F(B(R))$ be the closure of

$L^\infty(B(R))$ in $X_F(B(R))$. By [1, Thm. 8.21], the set $C_0^\infty(\mathbb{D})$ is dense in $E_F(\mathbb{D})$ with respect to the norm of X_F . Thus when μ is a non-smooth Beltrami coefficient satisfying the assumption (4.1) and $\varepsilon > 0$ we can find a smooth function $\theta \in C_0^\infty(\mathbb{D})$, $\|\theta\|_\infty < 2$ such that $\|\mu - \theta\|_F < \varepsilon$. Then, since $|\mu - \theta|$ is supported in \mathbb{D} and bounded by 3, we have

$$\begin{aligned} \|\mu \circ \psi_{\lambda_n}(\cdot, k_n) - \theta \circ \psi_{\lambda_n}(\cdot, k_n)\|_{L^s(\mathbb{C})}^s &= \int_{\mathbb{D}} |\mu(z) - \theta(z)|^s J_{g_n}(z) dm(z) \\ &\leq 3^{s-1} \left(\int_{\mathbb{D}} F(|\mu(z) - \theta(z)|) dm(z) \right) \left(\int_{\mathbb{D}} G(J_{g_n}(z)) dm(z) \right) \end{aligned} \quad (4.57)$$

where g_n is the inverse of the function $\psi_{\lambda_n}(\cdot, k_n)$. Then,

$$\int_{\mathbb{D}} G(J_{g_n}) dm \leq C \|J_{g_n}\|_{X^{1,1}(B(2))}$$

and by (4.48), $\|J_{g_n}\|_{X^{1,1}(\mathbb{D})}$ is uniformly bounded in n . Using (4.57) and (5.16) we see that (4.56) holds for all μ satisfying the assumption (4.1) and thus claim of the theorem follows. \square

4.4. $\bar{\partial}$ equations in k plane. Let us consider a Beltrami coefficient $\mu \in B_{exp}^p(\mathbb{D})$ and approximate μ with functions μ_n supported in \mathbb{D} for which $\lim_{n \rightarrow \infty} \mu_n(z) = \mu(z)$ and $\|\mu_n\|_\infty \leq c_n < 1$, see e.g. (3.16). Let $f_\mu(\cdot, k) \in W_{loc}^{1,Q}(\mathbb{C})$ be the solution of the equations

$$\bar{\partial}_z f_\mu(z, k) = \mu(z) \overline{\partial_z f_\mu(z, k)}, \quad \text{for a.e. } z \in \mathbb{C}, \quad (4.58)$$

$$f_\mu(z, k) = e^{ikz} \left(1 + \mathcal{O}_k\left(\frac{1}{z}\right)\right) \quad \text{for } |z| \rightarrow \infty, \quad (4.59)$$

and $f_{\mu_n}(\cdot, k) \in W_{loc}^{1,Q}(\mathbb{C})$ be the solution of the similar equations to Beltrami coefficients μ_n and μ , cf. Lemma 3.4. Here $\mathcal{O}_k(h(z))$ means a function of (z, k) that satisfies $|\mathcal{O}_k(h(z))| \leq C(k)|h(z)|$ for all z with some constant $C(k)$ depending on $k \in \mathbb{C}$. Let

$$\varphi_\mu(z, k) = (ik)^{-1} \log(f_\mu(z, k)), \quad \varphi_{\mu_n}(z, k) = (ik)^{-1} \log(f_{\mu_n}(z, k)),$$

c.f. (3.3). Then by (3.5) we have

$$|\varphi_{\mu_n}(z, k)| \leq |z| + 3, \quad |\varphi_\mu(z, k)| \leq |z| + 3. \quad (4.60)$$

By the proof of Lemma 3.4 we see that by choosing subsequence of μ_n , $n \in \mathbb{Z}_+$, which we continue to denote by μ_n , we can assume that

$$\lim_{n \rightarrow \infty} \varphi_{\mu_n}(z, k) = \varphi_\mu(z, k), \quad \text{uniformly in } (z, k) \in B(R) \times \{k_0\}, \quad (4.61)$$

for all $R > 0$ and $k_0 \in \mathbb{C}$.

Let us write the solutions f_{μ_n} and f_μ as

$$\begin{aligned} f_{\mu_n}(z, k) &= e^{ik\varphi_{\mu_n}(z, k)} = e^{ikz} M_{\mu_n}(z, k), \\ f_\mu(z, k) &= e^{ik\varphi_\mu(z, k)} = e^{ikz} M_\mu(z, k). \end{aligned}$$

Similar notations are introduced when μ is replaced by $-\mu$ etc. Let

$$\begin{aligned} h_{\mu_n}^{(+)}(z, k) &= \frac{1}{2}(f_{\mu_n}(z, k) + f_{-\mu_n}(z, k)), & h_{\mu_n}^{(-)}(z, k) &= \frac{i}{2}(\overline{f_{\mu_n}(z, k)} - \overline{f_{-\mu_n}(z, k)}), \\ u_{\mu_n}^{(1)}(z, k) &= h_{\mu_n}^{(+)}(z, k) - ih_{\mu_n}^{(-)}(z, k), & u_{\mu_n}^{(2)}(z, k) &= -h_{\mu_n}^{(-)}(z, k) + ih_{\mu_n}^{(+)}(z, k). \end{aligned}$$

Then by (4.60), $h_{\mu_n}^{(+)}(z, k)$ and $h_{\mu_n}^{(-)}(z, k)$ are uniformly bounded for $(z, k) \in B(R) \times \mathbb{C}$. By (4.61), we can define the pointwise limits

$$\lim_{n \rightarrow \infty} h_{\mu_n}^{(\pm)}(z, k) = h_{\mu}^{(\pm)}(z, k), \quad \lim_{n \rightarrow \infty} u_{\mu_n}^{(j)}(z, k) = u_{\mu}^{(j)}(z, k), \quad j = 1, 2. \quad (4.62)$$

The above formulae imply

$$u_{\mu}^{(2)}(z, k) = iu_{-\mu}^{(1)}(z, k) \text{ and } u_{\mu}^{(1)}(z, k) = -iu_{-\mu}^{(2)}(z, k). \quad (4.63)$$

Moreover, for

$$\begin{aligned} \tau_{\mu_n}(k) &= \frac{1}{2}(\overline{t_{\mu_n}(k)} - \overline{t_{-\mu_n}(k)}), & \tau_{\mu_n}(k) &= \frac{1}{2}(\overline{t_{\mu_n}(k)} - \overline{t_{-\mu_n}(k)}), \\ t_{\pm\mu_n}(k) &= \frac{i}{2\pi} \int_{\partial\mathbb{D}} M_{\pm\mu_n}(z, k) dz, & t_{\pm\mu}(k) &= \frac{i}{2\pi} \int_{\partial\mathbb{D}} M_{\pm\mu}(z, k) dz \end{aligned}$$

we see using dominated convergence theorem that $\lim_{n \rightarrow \infty} t_{\mu_n}(k) = t_{\mu}(k)$ for all $k \in \mathbb{C}$, and hence

$$\lim_{n \rightarrow \infty} \tau_{\mu_n}(k) = \tau_{\mu}(k) \quad \text{for all } k \in \mathbb{C}. \quad (4.64)$$

Then, as $|\mu_n| \leq c_n < 1$ correspond to conductivities σ_n satisfying $\sigma_n, \sigma_n^{-1} \in L^\infty(\mathbb{D})$, we have by [7, formula (8.2)] the $\bar{\partial}$ -equations with respect to the k variables

$$\bar{\partial}_k u_{\mu_n}^{(j)}(z, k) = -i\tau_{\mu_n}(k) \overline{u_{\mu_n}^{(j)}(z, k)}, \quad k \in \mathbb{C}, \quad j = 1, 2, \quad (4.65)$$

see also Nachman [52, 53] for different formulation on such equations. For $z \in \mathbb{C}$ functions $u_{\mu_n}^{(j)}(z, \cdot)$, $n \in \mathbb{Z}_+$ are uniformly bounded, the limit (4.62) and the dominated convergence theorem imply that $u_{\mu_n}^{(j)}(z, \cdot) \rightarrow u_{\mu}^{(j)}(z, \cdot)$ as $n \rightarrow \infty$ in $L^p(B(R))$ for all $p < \infty$ and $R > 0$. Since functions $|\tau_{\mu_n}(k)|$, $n \in \mathbb{Z}_+$ are uniformly bounded in compact sets, the pointwise limits (4.62), (4.64) and the equation (4.65) yield that

$$\bar{\partial}_k u_{\mu}^{(j)}(z, k) = -i\tau_{\mu}(k) \overline{u_{\mu}^{(j)}(z)}, \quad k \in \mathbb{C}, \quad j = 1, 2 \quad (4.66)$$

holds for all $z \in \mathbb{C}$ in sense of distributions and $u_{\mu}^{(j)}(z, \cdot) \in W_{loc}^{1,p}(\mathbb{C})$ for all $p < \infty$.

4.5. Proof of uniqueness results for isotropic conductivities. Proof of Theorem 1.9. Let us consider isotropic conductivities σ_j , $j = 1, 2$. Due to the above proven results, the proof will go along the lines of Section 8 of [7], where L^∞ -conductivities are considered, and its reformulation presented in Section 18 of [6] in a quite straight forward way when the changes explained below are made. The key point is the following proposition.

Proposition 4.10. *Assume that $\mu \in B_{exp}^p(\mathbb{D})$ and let $f_{\pm\mu}(z, k)$ satisfy (4.58)-(4.59) with the Beltrami coefficients $\pm\mu$. Then $f_{\pm\mu}(z, k) = e^{izk}M_{\pm\mu}(z, k)$, where*

$$\operatorname{Re} \frac{M_{+\mu}(z, k)}{M_{-\mu}(z, k)} > 0 \quad (4.67)$$

for every $z, k \in \mathbb{C}$.

Proof. Let us consider the Beltrami coefficients $\mu_n(z)$, $n \in \mathbb{Z}_+$ defined Section 4.4 that converge pointwise to $\mu(z)$ and satisfy $|\mu_n| \leq c_n < 1$. By Lemma 3.2 the functions $M_{\pm\mu_n}(z, k)$ do not obtain value zero anywhere. By [7, Prop. 4.3], the inequality (4.67) holds for the functions $M_{\pm\mu_n}(z, k)$. Then, $f_{\pm\mu_n}(z, k) \rightarrow f_{\pm\mu}(z, k)$ as $n \rightarrow \infty$ for all $k, z \in \mathbb{C}$, and thus see that

$$\operatorname{Re} \frac{M_{+\mu}(z, k)}{M_{-\mu}(z, k)} = \lim_{n \rightarrow \infty} \operatorname{Re} \frac{M_{+\mu_n}(z, k)}{M_{-\mu_n}(z, k)} \geq 0. \quad (4.68)$$

To show that the equality does not hold in (4.68), we assume the opposite. In this case, there are z_0 and k_0 such that

$$M_{+\mu}(z_0, k_0) = itM_{-\mu}(z_0, k_0) \quad (4.69)$$

with some $t \in \mathbb{R} \setminus \{0\}$. Then

$$f(z, k_0) = e^{ik_0z}(M_{+\mu}(z, k_0) - itM_{-\mu}(z, k_0))$$

is a solution of (4.58) and satisfies the asymptotics

$$f(z, k_0) = (1 - it)e^{ik_0z}(1 + \mathcal{O}(\frac{1}{z})) \quad \text{for } |z| \rightarrow \infty.$$

By using (2.13) to write the equation (4.58) in the form (3.11) and applying Lemma 3.2, we see that the solution $f(z, k_0)$ can be written in the form

$$f(z, k_0) = (1 - it)e^{ik_0\varphi(z)}.$$

This is in contradiction with the assumption (4.69) that implies $f(z_0, k_0) = 0$ and proves (4.67). \square

Let $f_{\pm\mu}(z, k)$ be as in Prop. 4.10 and use below for the functions defined in (4.62) the short hand notation $u_\mu^{(1)}(z, k) = u_1(z, k)$ and $u_\mu^{(2)}(z, k) = u_2(z, k)$. Then $u_1(z, k)$ and $u_2(z, k)$ are solutions of the equation (4.66). A direct computation shows also that

$$\nabla \cdot \sigma \nabla u_1(\cdot, k) = 0 \quad \text{and} \quad \nabla \cdot \frac{1}{\sigma} \nabla u_2(\cdot, k) = 0,$$

where $\sigma(z) = (1 - \mu(z))/(1 + \mu(z))$ is the conductivity corresponding to μ . Note that the conductivity $1/\sigma(z) = (1 + \mu(z))/(1 - \mu(z))$ is the conductivity corresponding to $-\mu$.

Generally, the near field measurements, that is, the Dirichlet-to-Neumann map Λ_σ on $\partial\Omega$ determines the scattering measurements, in particular the scattered fields outside Ω , see Nachman [52]. In our setting this means that we can use Lemma 5.1 and argue, e.g. as in the proof of Proposition 6.1 in [7] to see, that Λ_σ determines uniquely the solutions $f_{\pm\mu}(z_0, k)$ and $\tau_{\pm\mu}(k)$ for $z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}$ and $k \in \mathbb{C}$. We note that a constructive method based on integral equations on $\partial\mathbb{D}$ to determine $f_{\pm\mu}(z_0, k)$ from Λ_σ is presented in [9].

As $u_j(z, \cdot)$, $j = 1, 2$ are bounded and non-vanishing functions which satisfy (4.66), we have $\bar{\partial}u_j(z, \cdot) \in L_{loc}^\infty(\mathbb{C})$. This implies that $\partial u_j(z, \cdot) \in \text{BMO}_{loc}(\mathbb{C}) \subset L_{loc}^p(\mathbb{C})$ for all $p < \infty$, see e.g. [6, Thm. 4.6.5], and hence $u_j(z, \cdot) \in W_{loc}^{1,p}(\mathbb{C})$.

Let us now consider the isotropic conductivities σ and $\tilde{\sigma}$ in $\Omega = \mathbb{D}$ which are equal to 1 near $\partial\mathbb{D}$ and satisfy (1.23). Assume that $\Lambda_\sigma = \Lambda_{\tilde{\sigma}}$. Then, by the above considerations, $\tau_{\pm\mu}(k) = \tau_{\pm\tilde{\mu}}(k)$ for $k \in \mathbb{C}$.

Let $\mu = (1 - \sigma)/(1 + \sigma)$ and $\tilde{\mu} = (1 - \tilde{\sigma})/(1 + \tilde{\sigma})$ be the Beltrami coefficients corresponding to σ and $\tilde{\sigma}$.

By applying Lemma 3.3 with $k = 0$ we see that $f_\mu(z, 0) = 1$ for all $z \in \mathbb{C}$ and hence $u_1(z, 0) = 1$. By Lemma 3.2, the map $z \mapsto f_\mu(z, k)$ is continuous. Thus $u_1 \in \mathcal{X}^p$, $1 < p < \infty$, where \mathcal{X}^p is the space of functions $v(z, k)$, $(k, z) \in \mathbb{C}^2$ for which $v(z, \cdot) \in W_{loc}^{1,p}(\mathbb{C})$ and $v(z, \cdot)$ are bounded for all $z \in \mathbb{C}$ and the function $v(\cdot, k)$ is continuous for all $k \in \mathbb{C}$. These properties are crucial in the following Lemma which is a reformulation of the properties of the functions $u_1(z, k)$, $z, k \in \mathbb{C}$ proven in [7] for L^∞ -conductivities.

Lemma 4.11. (i) *Functions $u_1(z, k)$ with $k \neq 0$ have the z -asymptotics*

$$u_1(z, k) = \exp(ikz + v(z; k)), \quad (4.70)$$

where $C(k) > 0$ is such that $|v(z, k)| \leq C(k)$ for all $z \in \mathbb{C}$.

(ii) *Functions $u_1(z, k)$ have the k -asymptotics*

$$u_1(z, k) = \exp(ikz + k\varepsilon_\mu(k; z)), \quad k \neq 0 \quad (4.71)$$

where for each fixed z we have $\varepsilon_\mu(k; z) \rightarrow 0$ as $k \rightarrow \infty$.

(iii) *Let $1 < p < \infty$. The $u_1(z, k)$ given in (4.62) is the unique function in \mathcal{X}^p such that $u_1(z, k)$ non-vanishing, $u_1(z, 0) = 1$ for all $z \in \mathbb{C}$, and satisfies the $\bar{\partial}$ -equation (4.66) with the asymptotics and (4.70) and (4.71).*

Proof. (i) Let us omit the (z, k) variables in some expressions and denote $u_1(z, k) = u_1$, $f_\mu(z, k) = f_\mu$ etc. By definition of u_1 ,

$$u_1 = \frac{1}{2} (f_\mu + f_{-\mu} + \overline{f_\mu} - \overline{f_{-\mu}}) = f_\mu \left(1 + \frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}} \right)^{-1} \left(1 + \frac{\overline{f_\mu} - \overline{f_{-\mu}}}{f_\mu + f_{-\mu}} \right), \quad (4.72)$$

where each factor non-vanishing by Prop. 4.10. Thus (4.59) yields (4.70).

(ii) Let $F_t(z, k) = e^{-it/2} (f_\mu(z, k) \cos \frac{t}{2} + i f_{-\mu}(z, k) \sin \frac{t}{2})$, $t \in \mathbb{R}$. Then

$$\begin{aligned} \overline{\partial}_z F_t(z, k) &= \mu(z) e^{-it} \overline{\partial}_z F_t(z, k), \quad \text{for } z \in \mathbb{C}, \\ F_t(z, k) &= e^{ikz} (1 + O_k(z^{-1})) \quad \text{as } z \rightarrow \infty. \end{aligned}$$

Thus $F_t(z, k) = \exp(k\varphi_\lambda(z, k))$ where $\lambda = e^{-it}$ and $\varphi_\lambda(z, k)$ solves (4.43). Note that $f_\mu(z, k) = \exp(k\varphi_{\lambda_0}(z, k))$ where $\lambda_0 = 1$. Then

$$\frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}} + e^{it} = \frac{2e^{it} F_t}{f_\mu + f_{-\mu}} = \frac{\exp(k\varphi_\lambda(z, k))}{\exp(k\varphi_{\lambda_0}(z, k))} \frac{2e^{it}}{1 + M_{-\mu}(z, k)/M_\mu(z, k)}. \quad (4.73)$$

By Theorem 4.9 we have for $z \in \mathbb{C}$ and $k \in \mathbb{C} \setminus \{0\}$

$$e^{-|k|\varepsilon_1(k)} \leq |M_{\pm\mu}(z, k)| \leq e^{|k|\varepsilon_1(k)}, \quad (4.74)$$

$$e^{-|k|\varepsilon_2(k)} \leq \inf_{|\lambda|=1} \left| \frac{\exp(k\varphi_\lambda(z, k))}{\exp(k\varphi_{\lambda_0}(z, k))} \right| \leq \sup_{|\lambda|=1} \left| \frac{\exp(k\varphi_\lambda(z, k))}{\exp(k\varphi_{\lambda_0}(z, k))} \right| \leq e^{|k|\varepsilon_2(k)} \quad (4.75)$$

where $\varepsilon_j(k) \rightarrow 0$ as $k \rightarrow \infty$. Since $\text{Re}(M_{-\mu}/M_\mu) > 0$, estimates (4.73), (4.74) and (4.75) yield for $z \in \mathbb{C}$, $k \neq 0$

$$\inf_{t \in \mathbb{R}} \left| \frac{f_\mu - f_{-\mu}}{f_\mu + f_{-\mu}} + e^{it} \right| \geq e^{-|k|\varepsilon(k)} \quad \text{and} \quad \frac{|f_\mu - f_{-\mu}|}{|f_\mu + f_{-\mu}|} \leq 1 - e^{-|k|\varepsilon(k)}.$$

This and (4.72) yield the k -asymptotics (4.71).

(iii) As observed above, the function $u_1(z, k)$ given in (4.62) satisfies the conditions stated in (iii).

Next, let $u_1(z, k)$ and $\tilde{u}_1(z, k)$ be two function which satisfy the assumptions of the claim. Let us consider the logarithms

$$\delta_1(z, k) = \log u_1(z, k), \quad \tilde{\delta}_1(z, k) = \log \tilde{u}_1(z, k), \quad k, z \in \mathbb{C}.$$

As $u_1(z, \cdot) \in W_{loc}^{1,p}(\mathbb{C})$ for some $p < \infty$ and $u_1(z, \cdot)$ is bounded and non-vanishing function, we see that $\delta_1(z, \cdot) \in W_{loc}^{1,p}(\mathbb{C})$. As $u_1(z, 0) = 1$, we have

$$\delta_1(z, 0) = 0, \quad \text{for } z \in \mathbb{C}. \quad (4.76)$$

Moreover, $z \mapsto \delta_1(z, k)$ is continuous for any k . Let $k \neq 0$ be fixed. Then by (4.70)

$$\delta_1(z, k) = ikz + v(z, k), \quad z \in \mathbb{C}, \quad (4.77)$$

where $v(\cdot, k)$ is bounded and we see using elementary homotopy theory [64] that the map $H_k : \mathbb{C} \rightarrow \mathbb{C}$, $H_k(z) = \delta_1(z, k)$ is a surjective.

The function $\tilde{\delta}_1(z, k)$ has the same above properties as $\delta_1(z, k)$. Next we want to show that $\delta_1(z, k) = \tilde{\delta}_1(z, k)$ for all $z \in \mathbb{C}$ and $k \neq 0$. As the map $H_k : z \mapsto \delta_1(z, k)$ is surjective for all $k \neq 0$, this follows if we show that

$$w \neq z \text{ and } k \neq 0 \Rightarrow \delta_1(w, k) \neq \tilde{\delta}_1(z, k). \quad (4.78)$$

For this end, let $z, w \in \mathbb{C}$, $z \neq w$. Functions u_1 and \tilde{u}_1 satisfy the same equation (4.66) with the coefficient $\tau(k) = \tau_\mu(k)$. Subtracting these equations from each other we see that the difference $g(k; w, z) = \delta_1(w, k) - \tilde{\delta}_1(z, k)$ satisfies

$$\begin{aligned} \bar{\partial}_k g(k; w, z) &= \gamma(k; w, z) g(k; w, z), \quad k \in \mathbb{C}, \\ \gamma(k; w, z) &= -i\tau(k) \exp(i \operatorname{Im} \delta_1(k; w, z)) E(i \operatorname{Im} g(k; w, z)), \end{aligned} \quad (4.79)$$

where $E(t) = (e^{-t} - 1)/t$. Here, $\gamma(\cdot; w, z)$ is a locally bounded function. As $w \neq z$, the principle of the argument for pseudo-analytic functions, see [7, Prop. 3.3], the equation (4.79), the boundedness of γ , and the asymptotics $g(k; w, z) = ik(w - z) + k\varepsilon(k, w, z)$, where $\varepsilon(k, w, z) \rightarrow 0$ as $k \rightarrow \infty$ imply that $k \mapsto g(k; w, z)$ vanishes for one and only one value of $k \in \mathbb{C}$. Thus by (4.76), $g(k; w, z) = 0$ implies that $k = 0$, and hence (4.78) holds. Thus $\delta_1(z, k) = \tilde{\delta}_1(z, k)$ and $u_1(z, k) = \tilde{u}_1(z, k)$ for all $z \in \mathbb{C}$ and $k \neq 0$. \square

Remark 4.1. Note that $\tau_{\pm\mu}(k)$ is determined by Λ_σ . Thus Lemma 4.11 means that $u_1(z, k)$ can be constructed as the unique complex curve $z \mapsto u_1(z, \cdot)$, $z \in \mathbb{C}$ in the space of the solutions of the $\bar{\partial}$ -equation (4.66) which has the properties stated in (iii).

When $u_j(z, k)$ and $\tilde{u}_j(z, k)$, $j = 1, 2$ are functions corresponding to μ and $\tilde{\mu}$, the above shows that $u_1(z, k) = \tilde{u}_1(z, k)$. Using $\tau_{-\mu}$ instead of τ_μ and (4.63), we see by Lemma 4.11 that $u_2(z, k) = \tilde{u}_2(z, k)$ for all $z \in \mathbb{C}$, and $k \neq 0$.

Thus $f_{\pm\mu}(z, k) = f_{\pm\tilde{\mu}}(z, k)$ for all $z \in \mathbb{C}$ and $k \neq 0$. By [6, Thm. 20.4.12], the Jacobians of $f_{\pm\mu} \in W_{loc}^{1,Q}(\mathbb{C})$ are non-vanishing almost everywhere. Thus we see using the Beltrami equation (4.58) and the fact that $f_{\pm\mu}(z, k) = f_{\pm\tilde{\mu}}(z, k)$ for all $z \in \mathbb{C}$ and $k \neq 0$ that $\mu = \tilde{\mu}$ a.e. Hence $\sigma = \tilde{\sigma}$ a.e. This proves the claim of Theorem 1.9. \square

5. REDUCTION OF THE INVERSE PROBLEM FOR AN ANISOTROPIC CONDUCTIVITY TO THE ISOTROPIC CASE

Let in this section, we assume that the weight function \mathcal{A} satisfies the almost linear growth condition (1.25). Let $\sigma = \sigma^{jk} \in \Sigma_{\mathcal{A}}(\mathbb{C})$ be a conductivity matrix for which $\sigma(z) = 1$ for z in $\mathbb{C} \setminus \Omega$ and in some neighborhood of $\partial\Omega$.

Let $z_0 \in \partial\Omega$, and define

$$\mathcal{H}_\sigma(z) = \int_{\eta_z} (\Lambda_\sigma(u|_{\partial\Omega}))(z') ds(z'), \quad (5.1)$$

where η_z is the path (oriented to positive direction) from z_0 to z along $\partial\Omega$. This map is called the σ -Hilbert transform, and it can be considered a bounded map

$$\mathcal{H}_\sigma : H^{1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)/\mathbb{C}.$$

As shown in beginning of Subsection 2.3, there exists a homeomorphism $F : \mathbb{C} \rightarrow \mathbb{C}$ such that $F(\Omega) = \tilde{\Omega}$, $\tilde{\sigma} = F_*\sigma$ is isotropic (i.e. scalar function times identity matrix), F and F^{-1} are $W^{1,P}$ -smooth, and $F(z) = z + O(1/z)$. Moreover, F satisfies conditions \mathcal{N} and \mathcal{N}^{-1} . Also, as σ is one near the boundary, we have that F and F^{-1} are C^∞ smooth near the boundary.

By definition of $\tilde{\sigma} = F_*\sigma$, we see that

$$\det(\tilde{\sigma}(y)) = \det(\sigma(F^{-1}(y))) \quad (5.2)$$

for $y \in \tilde{\Omega}$. Thus under the assumptions of Thm. 1.11 where $\det(\sigma), \det(\sigma)^{-1} \in L^\infty(\Omega)$ we see that the isotropic conductivity $\tilde{\sigma}$ satisfies $\tilde{\sigma}, \tilde{\sigma}^{-1} \in L^\infty(\Omega)$.

Let us next consider the case when assumptions of Theorem 1.8 are valid and we have $\mathcal{A}(t) = pt - p$, $p > 1$. Then, as F satisfies the condition \mathcal{N} , the area formula gives

$$\begin{aligned} I_1 &= \int_{\tilde{\Omega}} \exp(\exp(q(\tilde{\sigma}(y) + \frac{1}{\tilde{\sigma}(y)}))) dm(y) \\ &= \int_{\Omega} \exp(\exp(q(\det(\sigma(x))^{1/2} + \frac{1}{\det(\sigma(x))^{1/2}}))) J_F(x) dm(x). \end{aligned} \quad (5.3)$$

In the case when $\mathcal{A}(t) = pt - p$, $p > 1$ [5, Thm. 1.1] implies that $J_F \log^\beta(e + J_F) \in L^1(\Omega)$ for $0 < \beta < p$. Then, Young's inequality (5.18) with the admissible pair (4.50) implies that

$$\begin{aligned} &\int_{\Omega} \exp(\exp(q(\det(\sigma(x))^{\frac{1}{2}} + \frac{1}{\det(\sigma(x))^{\frac{1}{2}}})) J_F(x) dm(x) \\ &\leq \left(\int_{\Omega} \exp(\exp(\exp(q(\det(\sigma)^{\frac{1}{2}} + \frac{1}{\det(\sigma)^{\frac{1}{2}}}))) dm \right) \left(\int_{\Omega} (1 + J_F) \log(1 + J_F) dm \right) \end{aligned} \quad (5.4)$$

and if conductivity σ satisfies (1.21), we see that I_1 is finite for some $q > 0$.

Thus under assumptions of Theorem 1.8 we see that I_1 is finite for the isotropic conductivity $\tilde{\sigma}$.

Let $\rho = F|_{\partial\Omega}$. It follows from Lemma 2.4 and (2.28) that $\rho_*\Lambda_\sigma = \Lambda_{\tilde{\sigma}}$. Then, $\mathcal{H}_{\tilde{\sigma}}h = \mathcal{H}_\sigma(h \circ \rho^{-1})$ for all $h \in H^{1/2}(\partial\tilde{\Omega})$.

Next we seek for a function $G_\Omega(z, k)$, $z \in \mathbb{C} \setminus \Omega$, $k \in \mathbb{C}$ that satisfies

$$\bar{\partial}_z G_\Omega(z, k) = 0 \quad \text{for } z \in \mathbb{C} \setminus \bar{\Omega}, \quad (5.5)$$

$$G_\Omega(z, k) = e^{ikz} (1 + \mathcal{O}_k(\frac{1}{z})), \quad \text{as } z \rightarrow \infty, \quad (5.6)$$

$$\text{Im } G_\Omega(\cdot, k)|_{\partial\Omega} = \mathcal{H}_\sigma(\text{Re } G_\Omega(\cdot, k)|_{\partial\Omega}). \quad (5.7)$$

To study it, we consider a similar function $G_{\tilde{\Omega}}(\cdot, k) : \mathbb{C} \setminus \tilde{\Omega} \rightarrow \mathbb{C}$ corresponding to the scalar conductivity $\tilde{\sigma}$, which satisfies in the domain $\mathbb{C} \setminus \tilde{\Omega}$ the equations (5.5)-(5.6) and the boundary condition $\text{Im } G_{\tilde{\Omega}}(\cdot, k) = \mathcal{H}_{\tilde{\sigma}}(\text{Re } G_{\tilde{\Omega}}(\cdot, k))$ on $z \in \partial\tilde{\Omega}$. Below, let $\tilde{\mu} = (1 - \tilde{\sigma})/(1 + \tilde{\sigma})$ be the Beltrami coefficient corresponding to the conductivity $\tilde{\sigma}$.

Lemma 5.1. *Assume that $\sigma \in \Sigma_{\mathcal{A}}(\Omega)$ is one near $\partial\Omega$. Then for all $k \in \mathbb{C}$*

(i) *For $k \in \mathbb{C}$ and $z \in \mathbb{C} \setminus \tilde{\Omega}$ we have $G_{\tilde{\Omega}}(z, k) = W(z, k)$ where $W(\cdot, k) \in W_{loc}^{1,P}(\mathbb{C})$ is the a unique solution of*

$$\bar{\partial}_z W(z, k) = \tilde{\mu}(z) \overline{\partial_z W(z, k)}, \quad \text{for } z \in \mathbb{C}, \quad (5.8)$$

$$W(z, k) = e^{ikz} (1 + \mathcal{O}_k(\frac{1}{z})), \quad \text{as } z \rightarrow \infty. \quad (5.9)$$

(ii) *The equations (5.5)-(5.7) have a unique solution $G_{\Omega}(\cdot, k) \in C^\infty(\mathbb{C} \setminus \Omega)$ and $G_{\Omega}(z, k) = G_{\tilde{\Omega}}(F(z), k)$ for $z \in \mathbb{C} \setminus \Omega$.*

Proof. The definition of the Hilbert transform $\mathcal{H}_{\tilde{\sigma}}$ implies that any solution $G_{\tilde{\Omega}}(z, k)$ of (5.5)-(5.7) can be extended to a solution $W(z, k)$ of (5.8). On other hand, the restriction of the solution $W(z, k)$ of (5.8)-(5.9) satisfies (5.5)-(5.7). The equations (5.8)-(5.9) have a unique solution by Theorem 3.1. As the solution $W(\cdot, k)$ is analytic in $\mathbb{C} \setminus \text{supp}(\tilde{\sigma})$, the claim (i) follows.

The claim (ii) follows immediately as $F : \mathbb{C} \setminus \bar{\Omega} \rightarrow \mathbb{C} \setminus \tilde{\Omega}$ is conformal, $F(z) = z + \mathcal{O}(1/z)$, and $\mathcal{H}_{\tilde{\sigma}} h = \mathcal{H}_{\sigma}(h \circ \rho)$ for all $h \in H^{1/2}(\partial\tilde{\Omega})$. \square

Lemma 5.2. *Assume that Ω is given and that $\sigma \in \Sigma_{\mathcal{A}}(\Omega)$ is one near $\partial\Omega$. Then the Dirichlet-to-Neumann form Q_{σ} determines the values of the restriction $F|_{\mathbb{C} \setminus \Omega}$, the boundary $\partial\tilde{\Omega}$, and the Dirichlet-to-Neumann map $\Lambda_{\tilde{\sigma}}$ of the isotropic conductivity $\tilde{\sigma} = F_*\sigma$ on $\tilde{\Omega}$.*

Proof. When σ is identity near $\partial\Omega$, the Dirichlet-to-Neumann form Q_{σ} determines the Dirichlet-to-Neumann map Λ_{σ} . By Lemma 3.4 we have $W(z, k) = \exp(ik\varphi(z, k))$ where by Theorem 4.9

$$\lim_{k \rightarrow \infty} \sup_{z \in \mathbb{C}} |\varphi(z, k) - z| = 0. \quad (5.10)$$

As $G(z, k) = W(F(z), k)$ we have

$$\lim_{k \rightarrow \infty} \frac{\log G(z, k)}{ik} = \lim_{k \rightarrow \infty} \varphi(F(z), k) = F(z). \quad (5.11)$$

By Lemma 5.1 $F(z, k)$ can be constructed for any $z \in \mathbb{C} \setminus \Omega$ by solving the equations (5.5)-(5.9). Thus the restriction of F to $\mathbb{C} \setminus \Omega$ is determined by the

values of limit (5.11). As $\tilde{\Omega} = \mathbb{C} \setminus F(\mathbb{C} \setminus \Omega)$ and $\Lambda_{\tilde{\sigma}} = (F|_{\partial\Omega})_* \Lambda_{\sigma}$, this proves the claim. \square

Above we saw that if the assumptions of Theorem 1.8 for σ are satisfied then for the isotropic conductivity $\tilde{\sigma} = F_*\sigma$ we have $\tilde{\sigma}, \tilde{\sigma}^{-1} \in L^\infty(\tilde{\Omega})$. Also, under the assumptions of Theorem 1.8 for σ the integral I_1 in (5.3) is finite for some $q > 0$. Thus Theorems 1.8 and 1.11 follow by Theorem 1.9 and Lemma 5.2. \square

APPENDIX A: ORLICZ SPACES

For the proofs of the facts discussed in this appendix we refer to [1, 39].

Let $F, G : [0, \infty) \rightarrow [0, \infty)$ be bijective convex functions. The pair (F, G) is called a Young complementary pair if

$$F'(t) = f(t), \quad G'(t) = g(t), \quad g = f^{-1}.$$

In the following we will consider also extensions of these functions defined by $F, G : \mathbb{C} \rightarrow [0, \infty)$ by setting $F(t) = F(|t|)$ and $G(t) = G(|t|)$. By [39, Sec. I.7.4], there are examples of such pairs for which $F(t) = \frac{1}{p}t^p \log^a t$ and $G(t) = \frac{1}{q}t^q \log^{-a} t$ where $p, q \in (1, \infty)$, $p^{-1} + q^{-1} = 1$ and $a \in \mathbb{R}$. We define that $u : D \rightarrow \mathbb{C}$, $D \subset \mathbb{R}^2$ is in an Orlicz class $K_F(D)$ if

$$\int_{\mathbb{D}} F(|u(x)|) dm(x) < \infty. \quad (5.12)$$

The Orlicz space $X_F(D)$ is the smallest vector space containing the convex set $K_F(D)$.

For a Young complementary pair (F, G) one can define for $u \in X_F(D)$ the norm

$$\|u\|_F = \sup \left\{ \int_D |u(x)v(x)| dm(x) ; \int_D G(u(x)) dm(x) \leq 1 \right\}. \quad (5.13)$$

There is also a Luxemburg norm

$$\|u\|_{(F)} = \inf \left\{ t > 0 ; \int_D F\left(\frac{u(x)}{t}\right) dm(x) \leq 1 \right\} \quad (5.14)$$

which is equivalent to the norm $\|u\|_F$ and one always has

$$\|u\|_{(F)} \leq \|u\|_F \leq 2\|u\|_{(F)}. \quad (5.15)$$

By [1, Thm. 8.10], $L_X(D)$ is a Banach space with respect to the norm $\|u\|_{(F)}$. Moreover, it holds that (see [39, Thm. II.9.5 and II.10.5]),

$$\|u\|_{(F)} \leq 1 \quad \Rightarrow \quad \int_D F(u(x)) dm(x) \leq \|u\|_F, \quad (5.16)$$

$$\|u\|_{(F)} \geq 1 \quad \Rightarrow \quad \int_D F(u(x)) dm(x) \geq \|u\|_{(F)}. \quad (5.17)$$

We also recall the Young's inequality [39, Thm. II.9.3], $uv \leq F(u) + G(v)$ for $u, v \geq 0$ which implies

$$\left| \int_D u(x)v(x) dm(x) \right| \leq \|u\|_F \|u\|_G. \quad (5.18)$$

The set $K_F(D)$ is a vector space when F satisfies the Δ_2 condition, that is, there is $k > 1$ such that $F(2t) \leq kF(t)$ for all $t \in \mathbb{R}_+$, see [1, Lem. 8.8]. In this case $X_F(D) = K_F(D)$.

We will use functions

$$M_{p,q}(t) = |t|^p (\log(1 + |t|))^q, \quad 1 \leq p < \infty, \quad q \in \mathbb{R}$$

and use for $F(t) = M_{p,q}(t)$ the notations $X_F(D) = X^{p,q}(D)$ and $\|u\|_F = \|u\|_{X^{p,q}(D)}$. For $p = 2$ we denote $M_{2,q}(t) = M_q(t)$ and $X^{2,q}(D) = X^q(D)$. Note that if D is bounded, $1 < p < \infty$ and $0 < \varepsilon < p - 1$ then

$$L^{p+\varepsilon}(D) \subset X^{p,q}(D) \subset L^{p-\varepsilon}(D).$$

Finally, we note that the dual space of $X^q(D)$ is $X^{-q}(D)$ and

$$\left| \int_D u(x)v(x) dm(x) \right| \leq \|u\|_{X^q(D)} \|v\|_{X^{-q}(D)}. \quad (5.19)$$

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